The main goal of this class is to practice solving 2nd order, linear, homogeneous DE with constant coef. In particular, we will work several examples involving complex roots coming from the characteristic equation. The transformation from complex form of solution to real form of solution will be emphasized.

A key formula: Euler’s formula (identity)

\[ e^{i\theta} = \cos \theta + \sin \theta i \]
\[ e^{\alpha + \beta i} = e^{\alpha}(\cos \beta + \sin \beta i) \]

- Example 1. It is easy to see \( e^{-2 - \frac{\pi}{3} i} = e^{-2} (\cos \frac{-\pi}{3} + \sin \frac{-\pi}{3} i) = e^{-2} \left( \frac{1}{2} - \frac{\sqrt{3}}{2} i \right) \)

Variations of Euler’s formula

\[ e^{i\theta} = \cos \theta + \sin \theta i \] (1)
\[ e^{-i\theta} = \cos \theta - \sin \theta i \] (2)

Note: (2) is based on (1) and \( \cos(-\theta) = \cos \theta \), \( \sin(-\theta) = -\sin \theta \).

Note: for real numbers \( \alpha, \beta \), we call \( \alpha + i\beta \) and \( \alpha - i\beta \) complex conjugates. The above formulas actually confirm that \( e^{\alpha+i\beta} \) and \( e^{\alpha-i\beta} \) are also complex conjugates.

The sum of complex conjugate pair yields cosine and the difference of them yields sine. To this end, perform \( \frac{(1)+(2)}{2} \):

\[ \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos \theta \] (3)

Also, perform \( \frac{(1)-(2)}{2i} \):

\[ \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin \theta \] (4)

- Euler’s formula (1)–(4) provide transformation between complex exponential function and cos/sin function both ways.

- Example 2.

\[ \frac{e^{2xi} + e^{-2xi}}{2} = \cos(2x) \]
Example 3.

\[
\frac{e^{2xi} - e^{-2xi}}{2i} = \sin(2x)
\]

Example 4. Please work on this example in steps.

\[
\frac{e^{(3+2i)x} - e^{(3-2i)x}}{2i} = e^{3x}\sin(2x)
\]

Now, we are ready to solve a DE with complex roots and transform the solution into a real form (i.e. cos/sin form)

Example 5. \(y'' - 6y' + 13y = 0\)

Characteristic equation

\[
\lambda^2 - 6\lambda + 13 = 0
\]

\[
\lambda^2 - 6\lambda + 9 + 4 = 0
\]

\[
(\lambda - 3)^2 + 4 = 0
\]

\[
\lambda - 3 = \pm 2i
\]

Thus, two complex roots

\[
\lambda_1 = 3 + 2i, \quad \lambda_2 = 3 - 2i
\]

(5)

Two linearly independent solution (Wronskian \(W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \neq 0 \implies y_1, y_2 \text{ are linearly independent})

\[
y_1(x) = e^{(3+2i)x}, \quad y_2(x) = e^{(3-2i)x}
\]

So the general solution can be expressed as

\[
y(x) = C_1 e^{(3+2i)x} + C_2 e^{(3-2i)x}
\]

(6)

This is the complex form. To transform it into real form with cos/sin, we apply (3) to get the cos part, that is,

\[
\frac{y_1 + y_2}{2} = \frac{e^{(3+2i)x} + e^{(3-2i)x}}{2}
\]

\[
= \frac{e^{3x}e^{2ix} + e^{3x}e^{-2ix}}{2}
\]
\[ e^{3x} \left( \frac{e^{2ix} + e^{-2ix}}{2} \right) = e^{3x} \cos(2x) \quad \text{by (3)} \]

This is a linear combination of \( y_1, y_2 \), and therefore
\[ y_3 = e^{3x} \cos(2x) \quad \text{is a solution!} \]

What about a real form with sin? Well, it has been done in Example 4!
\[ \frac{y_1 + y_2}{2i} = \frac{e^{(3+2i)x} - e^{(3-2i)x}}{2i} = e^{3x} \sin(2x) \]

Thus,
\[ y_4 = e^{3x} \sin(2x) \quad \text{is another solution!} \]

We can then employ the Wronskian \( W[y_3, y_4] \neq 0 \) to show \( y_3, y_4 \) are linearly independent. Since we are solving a 2nd order DE, two is the right number of linearly independent solutions that are needed to construct general solution. In this case
\[ y_{\text{general}}(x) = D_1 y_3 + D_2 y_4 = D_1 e^{3x} \cos(2x) + D_2 e^{3x} \sin(2x) \]

or, in a prettier format
\[ y_{\text{general}}(x) = e^{3x} (D_1 \cos(2x) + D_2 \sin(2x)) \quad (7) \]

Observe the connection between the \textbf{complex roots} (5) of the characteristic equation and the \textbf{complex form} (6) of general solution and the \textbf{real form} (7) of general solution. The following theorem can be established

\textbf{Theorem.} Consider 2nd order, linear, homogeneous DE with constant coef
\[ ay'' + by' + cy = 0, \quad a \neq 0 \]

Suppose the two roots to the characteristic equation
\[ a\lambda^2 + b\lambda + c = 0 \]

are complex conjugates
\[ \lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta \]

Then, the complex form of general solution is
\[ y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \]

and the real form of general solution is
\[ y = e^{ax} \left( D_1 \cos(\beta x) + D_2 \sin(\beta x) \right) \]

Note that the coef inside cos/sin come from the imaginary part of \( \lambda_{1,2} \), and the coef inside the exponential comes from the real part of \( \lambda_{1,2} \).