1. (20’) The determinant of a 3-by-3 matrix can be computed fairly easily. Here is the wiki entry http://en.wikipedia.org/wiki/Determinant#3-by-3_matrices and this diagram is also useful http://en.wikipedia.org/wiki/File:Sarrus_rule.png

Now, consider matrix

\[ A = \begin{pmatrix}
3 & 0 & -3 \\
-3 & 6 & 0 \\
0 & -6 & 3 \\
\end{pmatrix} \]

(a) Find the characteristic equation \( \det(\lambda I - A) = 0 \) and show that the 3 eigenvalues are 0, \( 6 + 3i \), \( 6 - 3i \). Hint: the calculation can be easier if the characteristic equation is factorized before the \( \lambda \)'s are plugged in.

**Solution.**

\[
\lambda I - A = \begin{pmatrix}
\lambda - 3 & 0 & 3 \\
3 & \lambda - 6 & 0 \\
0 & 6 & \lambda - 3 \\
\end{pmatrix}
\]

So, its determinant

\[
\det(\lambda I - A) = (\lambda - 3)(\lambda - 6)(\lambda - 3) + (3)(3)(6) - 0 - 0
\]

\[ = \lambda^3 - 12\lambda^2 + 45\lambda + (-3)(-6)(-3) + (3)(3)(6) = \lambda(\lambda^2 - 12\lambda + 45) \]

Thus,

\[
\det(\lambda I - A) \implies \lambda = 0 \text{ or } \lambda^2 - 12\lambda + 45 = 0
\]

\[ \implies \text{eigenvalues are } 0, \frac{12 \pm \sqrt{(-12)^2 - 4(1)(45)}}{2} \]

which are simplified to 0, \( 6 \pm 3i \).

(b) Among the following 4 vectors, match 3 of them with the associated eigenvectors; and then, show that the remaining one is not an eigenvector.

\[
\begin{pmatrix}
-i \\
1 \\
i-1
\end{pmatrix}, \begin{pmatrix}
4 \\
2 \\
4
\end{pmatrix}, \begin{pmatrix}
4 \\
-2 \\
4
\end{pmatrix}, \begin{pmatrix}
i \\
1 \\
-i -1
\end{pmatrix}
\]

**Solution.** By definition of eigenvalue and eigenvector,

\[ A\vec{v} = \lambda \vec{v} \quad \text{where } \vec{v} \neq \vec{0} \text{ and } \lambda \text{ is a scalar} \]
So, we multiply $A$ from the left with the vectors above, and examine respectively if $A\vec{v}$ equals a scalar multiple of $\vec{v}$.

$$A\vec{v}_1 = A\begin{pmatrix} -i \\ 1 \\ i - 1 \end{pmatrix} = \begin{pmatrix} 3 - 6i \\ 6 + 3i \\ -9 + 3i \end{pmatrix}$$

Compare the second row of $A\vec{v}_1$, i.e. $6 + 3i$ with second row of $\vec{v}_1$, i.e. $1$ and we see that if $A\vec{v}_1$ is ever equal to a scalar multiple of $\vec{v}_1$, that scalar has to be $6 + 3i$. Then, we compute

$$(6 + 3i)\vec{v}_1 = (6 + 3i)\begin{pmatrix} -i \\ 1 \\ i - 1 \end{pmatrix} = \begin{pmatrix} 3 - 6i \\ 6 + 3i \\ -9 + 3i \end{pmatrix} = A\vec{v}_1.$$ 

So, yes! We proved $\vec{v}_1$ is an eigenvector associated with eigenvalue $6 + 3i$.

For $\vec{v}_4 = \begin{pmatrix} i \\ 1 \\ -i - 1 \end{pmatrix}$, it is exactly the complex conjugate of $\vec{v}_1$. Since $A$ is a real matrix, we know that its complex eigenvalues and eigenvectors appear in conjugate pairs. Thus, $\vec{v}_4 = \overline{\vec{v}_1}$ is an eigenvector associated with $6 - 3i = 6 + 3i$.

For $\vec{v}_2 = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$, we again use definition $A\vec{v} = \lambda \vec{v}$. First, compute

$$A\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \vec{v}_2$$

so $\vec{v}_2$ is an eigenvector associated with eigenvalue 0.

Finally, for $\vec{v}_3 = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$, we again use definition $A\vec{v} = \lambda \vec{v}$. First, compute

$$A\vec{v}_3 = \begin{pmatrix} 0 \\ -24 \\ 24 \end{pmatrix}$$
so there is NO way that \( A\vec{v}_3 \) equals some scalar multiply of \( \vec{v}_3 \). Therefore, \( \vec{v}_3 \) is not an eigenvector.

(c) Find 3 linearly independent solutions to

\[
\frac{d}{dt} \vec{x}(t) = A\vec{x}(t)
\]

in terms of complex numbers. Explain how would you verify their linear independence, but do not actually compute it!

\[
\vec{x}_1 = e^{(6+3i)t} \begin{pmatrix} -i \\ 1 \\ i-1 \end{pmatrix}, \quad \vec{x}_2 = e^{(6-3i)t} \begin{pmatrix} i \\ 1 \\ -i-1 \end{pmatrix}, \quad \vec{x}_3 = e^{0t} \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}
\]

I would verify that \( \vec{x}_1, \vec{x}_2, \vec{x}_3 \) are linearly independent by verifying the Wronskian

\[
W[\vec{x}_1, \vec{x}_2, \vec{x}_3] = \det (\vec{x}_1, \vec{x}_2, \vec{x}_3) = \det \begin{pmatrix} e^{(6+3i)t} & e^{(6-3i)t} & e^{0t} \\ -i & i & 4 \\ 1 & 1 & 2 \\ i-1 & -i-1 & 4 \end{pmatrix} \neq 0
\]

2. (20’) Find the particular solution, in terms of real numbers, for

\[
\begin{cases}
x_1' = -2x_1 + 4x_2, \quad x_1(0) = 0 \\
x_2' = -4x_1 - 2x_2, \quad x_2(0) = -1
\end{cases}
\]

Solution. \( \det(\lambda I - A) = (\lambda + 2)^2 - 4(-4) = (\lambda + 2)^2 + 16 = 0 \)

\[\Rightarrow \lambda_{1,2} = -2 \pm 4i.\]

For \( \lambda = -2 + 4i \), solve for \( \vec{v}_1 \) in

\[
(\lambda I - A)\vec{v}_1 = 0, \quad \text{i.e.} \quad \begin{pmatrix} 4i, & -4 \\ 4, & 4i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0
\]

\[\Rightarrow 4ia - 4b = 0, \quad \text{and} \quad 4a + 4ib = 0.\]

Only one equation above is linearly independent, so we use the first one \( 4ia - 4b = 0 \).

One of \( a, b \) is free to take any value, for example, \( a = 1 \). Then, plug \( a = 1 \) into the previous equation \( 4i1 - 4b = 0 \) \[\Rightarrow b = i.\]

Therefore, an eigenvector associated with \( \lambda_1 = 2 + 4i \) is

\[
\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}
\]
Note: \( \vec{v}_1 \) is not the only eigenvector associated with \( \lambda_1 \), but every other eigenvector associated with \( \lambda_1 \) should be some scalar multiple of this \( \vec{v}_1 \). For \( \lambda_2 = -2 - 4i \), it is the complex conjugate of \( \lambda_1 = -2 + 4i \). Since \( A \) is a real matrix, we know that \( \vec{v}_2 \) can be chosen as the complex conjugate of \( \vec{v}_1 \) as well

\[
\vec{v}_2 = \overline{\vec{v}_1} = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

Again, \( \vec{v}_2 \) is not the only eigenvector associated with \( \lambda_2 \).

In order to find the real form of the general solution, we begin with write down the complex form of a solution based on \( \lambda_1, \vec{v}_1 \),

\[
e^{\lambda_1 t} \vec{v}_1 = e^{(-2+4i)t} \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix}
\]

\[
= e^{-2t} \left( \cos 4t + i \sin 4t \right) \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix} \quad \text{by Euler's formula}
\]

\[
= e^{-2t} \left( \begin{array}{c} \cos 4t + i \sin 4t \\ i \cos 4t - \sin 4t \end{array} \right)
\]

Separate the real and complex parts of the above solution, we arrive at two linearly independent solutions in real form

\[
\text{Re}(e^{\lambda_1 t} \vec{v}_1) = e^{-2t} \begin{pmatrix} \cos 4t \\ -\sin 4t \end{pmatrix}, \quad \text{Im}(e^{\lambda_1 t} \vec{v}_1) = e^{-2t} \begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}
\]

Therefore the general solution can be written as

\[
\vec{x}(t) = c_1 e^{-2t} \begin{pmatrix} \cos 4t \\ -\sin 4t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}
\]

Finally, plug in the initial condition \( x_1(0) = 0, x_2(0) = -1 \), i.e. \( \vec{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \)

\[
c_1 e^{-2t} \begin{pmatrix} \cos 4t \\ -\sin 4t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix} \bigg|_{t=0} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\]

\[
\implies c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\]

\[
\implies c_1 = 0, \quad c_2 = -1
\]
Final answer is
\[ \vec{x}_{\text{particular}} = -e^{-2t} \begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix} \]

3. (20’) Problem 4 on Page 316 in the textbook is
\[
\begin{align*}
    x_1' &= 4x_1 + x_2 \\
    x_2' &= 6x_1 - x_2
\end{align*}
\]
(a) Find the general solution. You answer is not necessarily the same as given by the textbook on Page 548. Show all your work.

Solution. (...... after some computation ....) We have \( \lambda_1 = -2, \vec{v}_1 = \begin{pmatrix} 1 \\ -6 \end{pmatrix} \) and \( \lambda_2 = 5, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Then, the general solution is
\[ c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = \ldots \]

(b) Copy the graph on Page 548 and indicate which solution curve corresponds to
initial condition (1): \( \begin{align*}
    x_1(0) &= -4 \\
    x_2(0) &= 4
\end{align*} \)

and which curve corresponds to
initial condition (2): \( \begin{align*}
    x_1(0) &= 3 \\
    x_2(0) &= 3
\end{align*} \)

(c) Find the particular solution satisfying initial condition (2).

Solution. Use the general solution found in part (a),
\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \implies c_1 = 0, c_2 = 3 \]

So, \( \vec{x}(t) = 3e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

4. (20’) Problem 6 on Page 316 in the textbook is
\[
\begin{align*}
    x_1' &= 9x_1 + 5x_2 \\
    x_2' &= -6x_1 - 2x_2
\end{align*}
\]
(a) Find the general solution. You answer is not necessarily the same as given by the textbook on Page 548. **Show all your work.**

**Solution.** (...... after some computation ....) We have \( \lambda_1 = 3 \), \( \vec{v}_1 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} \) and \( \lambda_2 = 4 \), \( \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \). Then, the general solution is

\[
c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = ......
\]

(b) Copy the graph on Page 548 and indicate which solution curve corresponds to

initial condition (3): \[
\begin{align*}
x_1(0) &= 1 \\
x_2(0) &= 0
\end{align*}
\]

and which curve corresponds to

initial condition (4): \[
\begin{align*}
x_1(0) &= -1 \\
x_2(0) &= 0
\end{align*}
\]

Are these two curves symmetric in any sense?

**Solution.** Yes, they are symmetric about the origin. In other words, if you rotate one curve about the origin by 180 degrees, it coincides with the other curve.

(c) Find the particular solutions satisfying initial condition (3); then, find the particular solutions satisfying initial condition (4). Are these two solutions symmetric in any sense?

**Solution.** Use the general solution found in part (a), and plug in initial condition (3)

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies c_1 = -1, c_2 = 6
\]

So, the particular solution determined by (3) is \( \vec{x}(t) = -e^{3t} \vec{v}_1 + 6e^{4t} \vec{v}_2 \)

Use the general solution found in part (a), and plug in initial condition (3)

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies c_1 = 1, c_2 = -6
\]

So, the particular solution determined by (4) is \( \vec{x}(t) = e^{3t} \vec{v}_1 - 6e^{4t} \vec{v}_2 \)

These two particular solutions are symmetric in that they are exactly of opposite signs.
5. (20’) Problem 8 on Page 316 in the textbook is

\[
\begin{align*}
x_1' &= x_1 - 5x_2 \\
x_2' &= x_1 - x_2
\end{align*}
\]

(a) Find the general solution in terms of complex numbers.

**Solution.** Following the same steps as in Problem 2, we find

\[\lambda_1 = 2i, \quad \vec{v}_1 = \begin{pmatrix} 2i + 1 \\ 1 \end{pmatrix}, \quad \text{conjugate} \quad \lambda_2 = -2i, \quad \vec{v}_2 = \begin{pmatrix} -2i + 1 \\ 1 \end{pmatrix}.\]

General solution in complex form

\[
\vec{x}_{\text{complex}} = c_1 e^{2it} \begin{pmatrix} 2i + 1 \\ 1 \end{pmatrix} + c_2 e^{-2it} \begin{pmatrix} -2i + 1 \\ 1 \end{pmatrix}
\]

(b) Find the general solution in terms of real numbers.

**Solution.** Following the same steps as in Problem 2, we calculate

\[
e^{2it} \begin{pmatrix} 2i + 1 \\ 1 \end{pmatrix} = \left( \cos 2t + i \sin 2t \right) \begin{pmatrix} 2i + 1 \\ 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} (\cos 2t - 2 \sin 2t) + i(2 \cos 2t + \sin 2t) \\ \cos 2t + i \sin 2t \end{pmatrix}
\]

Separate the real and imaginary parts

\[
\text{real part} = \begin{pmatrix} \cos 2t - 2 \sin 2t \\ \cos 2t \end{pmatrix}, \quad \text{imaginary part} = \begin{pmatrix} 2 \cos 2t + \sin 2t \\ \sin 2t \end{pmatrix}
\]

and the general solution is a linear combination of these two solutions in real form.

**Note:** we can use different eigenvectors in part (a), which will lead to different linearly independent solutions to be used to make general solutions. For example, we could choose

\[\lambda_1 = 2i, \quad \vec{v}_1 = \begin{pmatrix} 5 \\ 1 - 2i \end{pmatrix}, \quad \text{conjugate} \quad \lambda_2 = -2i, \quad \vec{v}_2 = \begin{pmatrix} 5 \\ 1 + 2i \end{pmatrix}\]

and calculate

\[
e^{2it} \begin{pmatrix} 5 \\ 1 - 2i \end{pmatrix} = \left( \cos 2t + i \sin 2t \right) \begin{pmatrix} 5 \\ 1 - 2i \end{pmatrix}
\]

\[
= \begin{pmatrix} 5 \cos 2t + 5i \sin 2t \\ (\cos 2t + 2 \sin 2t) + i(\sin 2t - 2 \cos 2t) \end{pmatrix}
\]
The real and imaginary parts would be
\[
\begin{pmatrix}
5 \cos 2t \\
\cos 2t + 2 \sin 2t
\end{pmatrix}
\text{ and }
\begin{pmatrix}
5 \sin 2t \\
\sin 2t - 2 \cos 2t
\end{pmatrix}
\]

(c) Copy the graph on Page 548 and indicate which solution curve corresponds to

initial condition (5): \[
\begin{align*}
x_1(0) &= 0 \\
x_2(0) &= 2
\end{align*}
\]

and which curve corresponds to

initial condition (6): \[
\begin{align*}
x_1(0) &= -4 \\
x_2(0) &= 2
\end{align*}
\]