1. (40’) The suspension system of a car uses a spring-mass system to absorb impacts and dampers (or shock absorbers) to control spring motions. Suppose the spring follows the Hooke’s law of elasticity which means the displacement of the mass is in direct proportion with the load added to it

\[ F_{\text{elastic}} = -kx. \]

Let’s fix \( k = 25 \) throughout this problem. Also fix the mass \( m = 1 \).

(i) (7’) Introduce a damper in the suspension to add a drag force that is proportional to the velocity of the mass,

\[ F_{\text{drag}} = -\gamma \frac{dx}{dt}. \]

Here, \( \gamma \) is a positive constant and the negative sign above indicates the drag force always works against the motion of the mass submerged in it. Fix \( \gamma = 6 \). Establish a new DE and find the particular solution with the initial data \( x(0) = 0 \) and \( x'(0) = -20 \).

(ii) (3’) Plot \( x \) vs \( t \) with the particular solution from part (i). Estimate how long it takes for the amplitude of the oscillation to reduce by half.

(iii) (10’) The state-of-the-art suspension system allows the driver to change the coefficient of drag force \( \gamma \) in part (i). Now, what is the critical value \( \gamma_c \) such that, upon gearing the constant \( \gamma > \gamma_c \), the driver won’t feel oscillating motion at all after hitting a bump? — the catch, however, for having overdamped suspension is that the driver experiences large shock in relative short times.

(iv) (10’) Let us again fix the value of \( \gamma \) in (i). If an external force \( f(t) = 10 \sin(5t) \) is applied to the suspension, find the particular solution subject to the initial data \( x(0) = 0 \) and \( x'(0) = -20 \).

(v) (10’) With the same external force and \( \gamma \) value as in (iv), solve the DE with a different set of initial data \( x_1(0) = x_1'(0) = 0 \). Then, prove that \( x(t) \) from (iv) and \( x_1(t) \) from this case become asymptotically close if we wait long enough, that is

\[ \lim_{t \to \infty} (x(t) - x_1(t)) = 0 \]
2. (15’) This is the scaling law of Laplace transform. Let $k$ be a constant and let

$$F(s) = \mathcal{L}\{f(t)\}$$

for a generic function defined for $t > 0$. Use the definition of Laplace transform to prove that

$$\frac{1}{k} F\left(\frac{s}{k}\right) = \mathcal{L}\{f(kt)\}$$

3. (5×5’=25’) For this problem, you may selectively apply Theorem 1 on page 444, Theorem 1 on page 453, Corollary on page 454, Theorem 1 on page 465, Theorem 2 on page 476 and Theorem 1 on page 482.

Given the Laplace transform

$$\mathcal{L}\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$$

use the properties of Laplace transform and inverse Laplace transform to find

(i) $$\mathcal{L}\left\{\frac{2}{\sqrt{\pi t}} - \frac{1}{\sqrt{2t}}\right\}$$

(ii) $$\mathcal{L}\left\{\frac{d^2}{dt^2} \frac{1}{\sqrt{\pi t}}\right\}$$

(iii) $$\mathcal{L}\left\{e^{-4t} \cdot \frac{1}{\sqrt{\pi t}}\right\}$$

(iv) $$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{8s}}\right\}$$

(v) $$\mathcal{L}^{-1}\left\{\frac{5}{\sqrt{s-5}}\right\}$$

4. (20’) Find $\mathcal{L}^{-1}\left\{\frac{8}{(s^2 - s - 2)(s^2 + 4)}\right\}$. Show every step.