MAT 243 (Meets: 3:30pm – 4:45pm MW)

Test 2

Closed Book. Calculator is ok, but won’t be necessary.

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Name: ______________________________  ASU ID: ______________________________

Score: _____________________________/100'

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Honor Statement

By signing below, I confirm that I have never (and will never) given or received any
unauthorized assistance on this exam.

Signature: ___________________________  Date: ___________________________

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• Informal proofs will be fine.

• Use the back sides of the sheets if it is necessary.
1. (25 points = 3’+5’+5’+5’+7’) Note: to answer “Why or why not?”, an informal proof will be fine.

Let function \( f(x) = [2x] \) and \( g(x) = \sqrt{x-4} \)

(e) Let \( f(x) \) be a mapping from \( R \) to \( Z \). Find \( f([-1.1, 2.2]) \) and \( f^{-1} \{ -1 \} \).

**Solution.** The image of a floor function is always an integer. So,

\[
f([-1.1, 2.2]) = [-3, 4] \cap Z = \{-3, -2, -1, 0, 1, 2, 3, 4\}
\]

2. (20 points = 5’+10’+5’) Sequences and Summations.

(b) Simplify the following summation to an expression in terms of \( n \) without the \( \sum \) sign or \( j \),

\[
\sum_{j=n}^{2n} (3j + (-3)^j - 3)
\]

**Solution.** Separate the sum into arithmetic and geometric progressions,

\[
\sum_{j=n}^{2n} (3j - 3) + \sum_{j=n}^{2n} (-3)^j
\]

Then, employ the formulas for each one of them,

**sum of arithmetic prog.** = (number of terms)*(first term + last term)/2

\[
\Rightarrow \sum_{j=n}^{2n} (3j - 3) = (n + 1) * (3n - 3 + 6n - 3)/2 = \frac{1}{2}(n + 1)(9n - 6).
\]

And,

**sum of geometric prog.** = \( \frac{\text{first term - one term after the last}}{1 - \text{common ratio}} \)

\[
\Rightarrow \sum_{j=n}^{2n} (-3)^j = \frac{(-3)^n - (-3)^{2n+1}}{1 - (-3)} = \frac{1}{4}((-3)^n - (-3)^{2n+1}).
\]

So, the answer is

\[
\sum_{j=n}^{2n} (3j + (-3)^j - 3) = \frac{1}{2}(n + 1)(9n - 6) + \frac{1}{4}((-3)^n - (-3)^{2n+1}).
\]
4. (25 points = 10’ + 10’ + 5’) Complexity.

(a) Proof that if both \( f(x) \) and \( g(x) \) are \( O(x^n) \), then \( f(x)g(x) \) is \( O(x^{2n}) \). Note: \( O \) relation does not necessarily imply the existence of certain limit.

Solution. By definition, \( f(x) \) is \( O(x^n) \) means
\[
\exists \text{ positive numbers } k_1 \text{ and } C_1 \text{ so that } |f(x)| < C_1 x^n \text{ for any } x > k_1.
\]
Similarly, \( g(x) \) is \( O(x^n) \) means
\[
\exists \text{ positive numbers } k_2 \text{ and } C_2 \text{ so that } |g(x)| < C_2 x^n \text{ for any } x > k_2.
\]
So, multiply the above inequalities,
\[
|f(x)g(x)| < C_1 C_2 x^n \cdot x^n \text{ for any } x > k_1 \text{ and } x > k_2,
\]
namely, \( f(x)g(x) \) is \( O(x^{2n}) \) with witnesses \( C = C_1 C_2 \text{ and } k = \max\{k_1, k_2\} \).

(b) Prove that \( 100n^2 \log n + 2n^3 \) is \( O(n^3) \) using definitions or limits. You may use the fact that \( n > \log n \) for any \( n \geq 1 \). The limits, if ever used, have to be proved with rigor.

• If we use the definitions, then it goes like,
\[
n > \log n \text{ for any } n \geq 1 \implies 100n^2 \log n < 100n^2 n \text{ for any } n \geq 1
\]
therefore, \( 100n^2 \log n + 2n^3 < 100n^2 n + 2n^3 = 102n^3 \) for any \( n \geq 1 \).
This shows that \( 100n^2 \log n + 2n^3 \) is \( O(n^3) \) with witnesses \( C = 102, k = 1 \).

• If we use limits, then it goes like,
\[
\lim_{n \to \infty} \frac{100n^2 \log n + 2n^3}{n^3} = \lim_{n \to \infty} \left( \frac{100n^2 \log n}{n^3} + \frac{2n^3}{n^3} \right) = \lim_{n \to \infty} \frac{100 \log n}{n} + 2
\]
\[
= \lim_{n \to \infty} \frac{100 \frac{\log n}{n}}{1} + 2
\]
(note: since both \( 100 \log n \text{ and } n \) approach \( \infty \), we can apply the l’Hopital rule here.)
\[
= 0 + 2 = 2.
\]
The limit of the ratio is a finite number, thus the big \( O \) relations holds.