GEOMETRICAL TOOLS FOR PDES ON A SURFACE

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Abstract. This is an excerpt from my paper with A. Mahalov [1]. PDE theories with Riemannian geometry are long studied subject — c.f. for example the texts cited in the bibliography. Here, I only present a very brief and elementary explanation, mainly about differential operators on a 2D surface. Generalization to higher dimension needs more caution.

Let \( M \) denote a 2-dimensional, compact, Riemannian manifold without boundary, typically the unit sphere \( M = S^3 - \{1\} \), endowed with metric \( g \) inherited from the embedding Euclidean space \( \mathbb{R}^3 \). Let \( p \in M \) denote a point with local coordinates \((p_1, p_2)\).

Any vector field \( u \) in the tangent bundle \( TM \) is identified with a field of directional differentials, which is written in local coordinates as

\[
    u = \sum_i a^i \frac{\partial}{\partial p_i}.
\]

We use the notation

\[
    \nabla_u f := \sum_i a^i \frac{\partial f}{\partial p_i} \tag{0.1}
\]

to denote the directional derivative of a scalar-valued function \( f \) in the direction of \( u \). Using the orthogonal projection \( \text{Proj}_{\mathbb{R}^3 \to TM} \) induced by the Euclidean metric of \( \mathbb{R}^3 \), we define the covariant derivative of a vector field \( v \in TM \) along another vector field \( u \in TM \),

\[
    \nabla_u v := \text{Proj}_{\mathbb{R}^3 \to TM} \sum_{i=1}^3 (\nabla_u v_i) e_i. \tag{0.2}
\]

Here, \( v \) is expressed in an orthonormal basis of \( \mathbb{R}^3 \) as \( v = v_1 e_1 + v_2 e_2 + v_3 e_3 \).

The metric \( g \) is identified with a \((0, 2)\) tensor, simply put, an \( 2 \times 2 \) matrix \((g_{ij})_{2 \times 2}\) in local coordinates. Thus, the vector inner product follows

\[
    g\left( \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right) = g_{ij} \quad \text{for } 1 \leq i, j \leq 2.
\]

The definitions and basic properties of differential forms can be found in e.g. [4]. We only sketch the following facts that will be used in this section. A differential \( k \)-form \( \beta \), at given point \( p \in M \), maps any \( k \)-tuple of tangent vectors to a scalar. In particular, a \( 0 \)-form is identified with a scalar-valued function. The 1-form \( dp_i \) in local coordinates satisfies \( dp_i \left( \frac{\partial}{\partial p_j} \right) = \delta_{ij} \). The exterior differential \( d \) maps a \( k \)-form to a \( (k + 1) \) form. For example, for \( 0 \)-form \( f \), \( df = \sum_i \frac{\partial f}{\partial p_i} dp_i \) so that \( df(u) = \nabla_u f \). The wedge product of a \( k \)-form \( \alpha \) and \( l \)-form \( \beta \), denoted by \( \alpha \wedge \beta \), is a \((k + l)\) form. It is skew-commutative in the sense that \( \alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \).

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0.1. Hodge Theory. ([4, 3])

The Hodge *-operator, defined in an orthonormal basis $\frac{\partial}{\partial p_1}$, $\frac{\partial}{\partial p_2}$ in a sub-region of $M$, satisfies

$$^*dp_1 = dp_2, \quad ^*dp_2 = -dp_1, \quad ^*1 = dp_1 \wedge dp_2, \quad *(dp_1 \wedge dp_2) = 1.$$ 

It is easy to see that Hodge *-operator maps between $k$-forms and $(n-k)$form. And its square, ** amounts to identity or (-1) times identity.

Using the Hodge star operator, we define the co-differential for any $k$-forms $\alpha$ in an $n$-dimensional manifold,

$$\text{codifferential} : \quad \delta \alpha := (-1)^k \cdot ^*d \cdot \alpha = (-1)^{n(k+1)+1} \cdot d \cdot \alpha,$$

and in particular, for $n = 2$,

$$\delta \alpha = -\cdot d \cdot \alpha.$$

So, $\delta$ maps a $k$-form to a $(k-1)$form.

The Hodge Laplacian (a.k.a. Laplace-Beltrami operator and Laplace-de Rham operator) is then defined by

$$\Delta_H := d \delta + \delta d. \quad (0.3)$$

In particular, for a scalar-valued function $f$ in a local basis $\left\{ \frac{\partial}{\partial p_i} \right\}$ with metric $g$, it is identified as

$$\Delta_H f = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_i(\sqrt{|g|}g^{ij} \partial_j f)$$

where $(g^{ij})$ is the matrix inverse of $(g_{ij})$. Thus, on a surface $M$, the Hodge Laplacian $\Delta_H$ defined in $(0.3)$ amounts to the surface Laplacian $\Delta_M$ times $(-1)$. In particular, if $M$ is a two-dimensional surface, then

for scalar function $f$, \quad $\Delta_M f = -\delta df = ^*d \cdot df \quad (0.4)$

since $\delta f = 0$ for a 0-form $f$. For consistency, we also fix the surface Laplacian $\Delta_M$ of 1-forms as the Hodge Laplacian $\Delta_H$ times $(-1)$,

for 1-form $\alpha$, \quad $\Delta_M \alpha = -(d \delta + \delta d) \alpha = (d \cdot d + ^*d \cdot d) \alpha \quad (0.5)$

For now on, we will use $\Delta$ for $\Delta_M$.

The Hodge decomposition theorem in its most general form states that for any $k$-form $\omega$ on an oriented compact Riemannian manifold, there exist a $(k-1)$-form $\alpha$, $(k+1)$-form $\beta$ and a harmonic $k$-form $\gamma$ satisfying $\Delta_H \gamma = 0$, s.t.

$$\omega = d \alpha + \delta \beta + \gamma.$$ 

In particular, for any 1-form $\omega$ on a 2-dimensional manifold with the 1st Betti number 0 (loosely speaking, there is no “holes”), there exist two scalar-valued functions $\Phi$, $\Psi$ such that

$$\omega = d \Phi + \delta(\ast \Psi) = d \Phi - \ast d \Psi. \quad (0.6)$$

Here, the third term drops out of the RHS of $(0.6)$ because, by the Hodge theory, the dimension of the space of harmonic $k$-forms on $M$ equals the $k$-th Betti number of $M$. In the cohomology class containing the unit sphere $S^2$, the 0th, 1st and 2nd Betti numbers are respectively 1, 0, 1. In other words, the only harmonic 1-form on $S^2$ is zero.

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1The existence of such basis is guaranteed by the Gram-Schmidt orthogonalization process.
In connection with vector fields. In a Riemannian manifold, there is a 1-to-1 correspondence, induced by the metric \( g \), between vectors and 1-forms. They are the so called “musical isomorphisms” denoted by \( \flat \) and \( \sharp \). For any vector fields \( u, v \), the 1-form \( u^\flat \) satisfies,

\[
\begin{align*}
u^\flat(v) &= g(u, v), \\
(u^\flat)^\sharp &= u.
\end{align*}
\]

In a (local) orthonormal basis, \( \flat \) and \( \sharp \) map between vectors and 1-forms with identical coordinates.

In a 2-dimensional Riemannian manifold, in order to define the divergence and curl of a vector field \( u \in T^1 \), we use \( \flat \) to map it to 1-form and then apply \( \delta \) and \( d \) to obtain the scalar fields

\[
\text{divergence, } \quad \text{div } u := -\delta (u^\flat) = *d (u^\flat)
\]

\[
\text{curl, } \quad \text{curl } u := -\delta (*u^\flat) = -*d (u^\flat)
\]

For a scalar field \( f \), we define gradient and its \( \pi/2 \) rotation as

\[
\text{gradient, } \quad \nabla f := (df)^\sharp
\]

\[
\text{rotated gradient, } \quad \nabla^\perp f := -(df)^\sharp
\]

We also define the counterclockwise \( \pi/2 \) rotation operator \( \perp \) acting on a vector field as

\[
u^\perp := -(u^\flat)^\sharp
\]

so that, consistently, \( \nabla^\perp f = (\nabla f)^\perp \) and \( \text{div } u = \text{curl } u^\perp \).

Combine these definitions with that of the surface Laplacian (0.4), (0.5) to obtain

\[
\text{Laplacian of scalar, } \quad \Delta f := \text{div } \nabla f = \text{curl } \nabla^\perp f
\]

\[
\text{Laplacian of vector, } \quad \Delta u := \nabla \text{div } u + \nabla^\perp \text{curl } u.
\]

An immediate consequence is that

\[
\Delta \text{ commutes with each one of div, curl, } \nabla, \nabla^\perp.
\]

To this end, the vector-field version of Hodge decomposition (0.6) becomes, for smooth vector field \( u \) on \( \mathbb{S}^2 \), there exist scalar fields \( \Phi, \Psi, \) s.t.

\[
\text{Hodge decomposition, } \quad u = \nabla \Phi + \nabla^\perp \Psi.
\]

We note that, by the virtue of (0.12), the decomposition satisfies

\[
\text{div } u = \Delta \Phi, \quad \text{curl } u = \Delta \Psi.
\]

It is also easy to use these definitions to verify the following properties,

\[
\text{curl } \nabla f = \text{div } \nabla^\perp f = 0
\]

due to \( dd = 0 \) and \( \delta \delta = 0 \); and,

\[
\text{curl } (fu^\perp) = \text{div } (fu) = \nabla f \cdot u + f \text{div } u,
\]

as a consequence of the product rule for differential \( d \).

\[\text{\footnote{Since 0-forms are identified with scalar fields, we use Hodge } ^\ast\text{-operator to map between 0-forms and 2-forms.}}\]
0.3. **In connection with surface integrals.** The integral of scalar field $f$ over an $n$-dimensional, oriented Riemannian manifold $M$, defined in differential-geometric terms (e.g. [4, Section 4.10]), is the integral of the $n$-form $\ast f$

$$\int_M f = \int_M \ast f. \quad (0.18)$$

Note that any $n$-form defines a measure on $M$.

For the most general case, the inner product of $k$-forms $\alpha_1, \alpha_2$ is defined as

$$\langle \alpha_1, \alpha_2 \rangle := \int_M (\alpha_1 \wedge \ast \alpha_2).$$

In particular, for vector fields $u, v$, we map them to 1-forms and define $L^2(M)$ inner product as

$$\langle u, v \rangle_{L^2(M)} := \langle u^\flat, v^\flat \rangle.$$

This coincides exactly with the more conventional definition

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta \beta \rangle.$$

It then follows from the celebrated Stokes theorem (e.g. [4]) that, in the case when $M$ has no boundary, the codifferential is the adjoint of exterior differential w.r.t. $L^2(M)$ inner product

$$\langle da, \beta \rangle = \langle a, \delta \beta \rangle.$$

This duality relation, together with definitions (0.7) – (0.13), immediately leads to the following “integrating-by-parts” formulas on a surface $M$, which generalize the Green’s identities and give adjoint relations of the differential operators we just defined.

$$\langle \nabla f, u \rangle = - \langle f, \text{div} \, u \rangle, \quad (0.19)$$

$$\langle \nabla \perp f, u \rangle = - \langle f, \text{curl} \, u \rangle, \quad (0.20)$$

$$\langle f, \Delta h \rangle = \langle \Delta f, h \rangle = - \langle \nabla f, \nabla h \rangle, \quad (0.21)$$

$$\langle u, \Delta v \rangle = \langle \Delta u, v \rangle = - \langle \text{div} \, u, \text{div} \, v \rangle - \langle \text{curl} \, u, \text{curl} \, v \rangle. \quad (0.22)$$

Here, for simplicity, the $L^2(M)$ subscript is omitted from all $\langle \cdot, \cdot \rangle$ expressions. The $\langle \cdot, \cdot \rangle$ of scalar fields is also easily understood.

0.4. **Local expression in terms of spherical coordinates for $M = S^2$.** Although the proofs in this article are independent of any local coordinate systems, we provide here, for interested readers, the spherical-coordinate forms of some of the differential operators defined above.

Let $\phi$ denote the longitude and $\theta$ the colatitude of a point on a sphere. Let $e_\phi, e_\theta$ denote the unit tangent vectors in the increasing directions of $\phi$ and $\theta$. Then, at point $p$ away from the poles,

$$\partial_\phi = \sin \theta e_\phi, \quad \partial_\theta = e_\theta,$$

namely,

$$\frac{1}{\sin \theta} \partial_\phi \quad \text{and} \quad \partial_\theta \quad \text{form an orthonormal basis of $T_M p$.}$$

Therefore, the musical isomorphisms, in $\phi, \theta$ coordinates, satisfy

$$\left( \frac{1}{\sin \theta} \partial_\phi \right)^\flat = \sin \theta d\phi \quad \text{and} \quad \left( \partial_\theta \right)^\flat = d\theta \quad \text{form an orthonormal basis of $T^* M_p$.}$$
In this context, the Hodge *-operator satisfies, with smooth scalar fields \( f_1, f_2, f \),

for 1-forms, \( * (f_1 d\phi + f_2 d\theta) = \frac{f_1}{\sin \theta} d\theta - f_2 \sin \theta d\phi \)

for 0-forms and 2-forms, \( * (f d\phi \wedge d\theta) = \frac{f}{\sin \theta} \), \( * f = \frac{f}{\sin \theta} \sin \theta d\phi \wedge d\theta \)

Combining the last equation with (0.18), we have the spherical expression for the integral of scalar field \( f \) over \( S^2 \),

\[
\int_{S^2} f = \int_0^\pi \int_0^{2\pi} f(\theta, \phi) \sin \theta d\phi d\theta,
\]

and inner product of vector fields, \( u = u_1 e_\phi + u_2 e_\theta \), \( v = v_1 e_\phi + v_2 e_\theta \),

\[
\langle u, v \rangle_{L^2(S^2)} = \int_0^\pi \int_0^{2\pi} (u_1 v_1 + u_2 v_2) \sin \theta d\phi d\theta
\]

The differential operators defined in (0.7) — (0.13) then become,

for scalar field \( f \)

\[
\nabla f = \frac{1}{\sin \theta} \partial_\phi f e_\phi + \partial_\theta f e_\theta
\]
\[
\nabla^\perp f = \partial_\phi f e_\phi - \frac{1}{\sin \theta} \partial_\theta f e_\theta
\]
\[
\Delta f = \frac{1}{\sin^2 \theta} (\partial^2_\phi f + \sin \theta \partial_\theta (\sin \theta \partial_\theta f))
\]

and

for vector field \( u = u_1 e_\phi + u_2 e_\theta \)

\[
\text{div } u = \frac{1}{\sin \theta} (\partial_\phi u_1 + \partial_\theta (u_2 \sin \theta))
\]
\[
\text{curl } u = \frac{1}{\sin \theta} (\partial_\theta u_2 - \partial_\phi (u_1 \sin \theta)).
\]

The surface Laplacian of \( u \) can also be expressed using (0.13) and the formulas above.

The directional derivative of a scalar, (0.1), can be expressed as

\[
\nabla_u f = \frac{1}{\sin \theta} u_1 \partial_\phi f + u_2 \partial_\theta f
\]

and the covariant derivative \( \nabla_u v \) can be expressed accordingly using the above formula and (0.2).

References


