MAT 194/294/394/494, Fall Semester 2006
Functional Equations

Functions from the Integers to the Integers

(A1) Let \( f \) be a function from \( \mathbb{Z}^+ \) to \( \mathbb{Z}^+ \). Prove that if
\[
f(n + 1) > f(f(n))
\]
for all \( n \in \mathbb{Z}^+ \), then \( f(n) = n \) for all \( n \).

(A2) Prove that \( f(n) = 1 - n \) is the only function from \( Z \) to \( Z \) that satisfies the following conditions:

(i) \( f(f(n)) = n \), for all integers \( n \).
(ii) \( f(f(n + 2) + 2) = n \), for all integers \( n \).
(iii) \( f(0) = 1 \).

(A3) Let \( f \) be defined on the natural numbers as follows: \( f(1) = 1 \) and for \( n > 1 \),
\[
f(n) = f(f(n - 1)) + f(n - f(n - 1)).
\]
Find, with proof, a simple explicit expression for \( f(n) \) which is valid for all \( n = 1, 2, \ldots \)

(A4) Determine all functions \( f \) from \( Z \) to \( Z \) satisfying
\[
f(x^3 + y^3 + z^3) = (f(x))^3 + (f(y))^3 + (f(z))^3
\]
for all integers \( x, y, \) and \( z \).

(A5) Suppose \( f \) is a function from \( \mathbb{Z}^* \) to \( \mathbb{Z}^* \), \( f(1) > 0 \), and
\[
f(m^2 + n^2) = f(m)^2 + f(n)^2
\]
for all integers \( m \) and \( n \). Show that \( f(n) = n \) for all \( n \).

Functions from the Reals to the Reals

(B1) Let \( f \) be a function from \( R \) to \( R \) such that, for some constant \( a \),
\[
f(x + a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}
\]
for all \( x \). Prove that \( f \) is periodic, and give an example of a non-constant function \( f \) that satisfies the equation above with \( a = 1 \).

(B2) Suppose \( f \) is a function from \( R \) to \( R \), and that
\[
f(x + y) = f(x) + f(y)
\]
for all reals \( x, y \). Show that \( f(x) = cx \) for some real number \( c \).
(B3) Suppose $f$ is a continuous function from $R^+$ to $R$, and that $f(1) = 5$ and

$$f\left(\frac{x}{x+1}\right) = f(x) + 2,$$

for all positive reals $x$.

(i) Find $\lim_{x \to +\infty} f(x)$.

(ii) Prove that $\lim_{x \to 0^+} f(x) = +\infty$.

(iii) Find all functions $f$ which satisfy the given conditions.

(B4) Suppose that $f$ is a function from $R$ to $R$, that

$$f(tx) = tf(x)$$

for all real $x$ and all nonnegative real $t$, and that $f$ is differentiable at 0. Prove that $f$ is linear.

(B5) Find every function $f$ from $R$ to $R$ such that

$$f(x^2 + y + f(y)) = 2y + (f(x))^2$$

for all reals $x$ and $y$.

(B6) Find every function $f$ from $R$ to $R$ that is continuous at 0, and satisfies

$$f(x + 2f(y)) = f(x) + y + f(y)$$

for all reals $x$ and $y$.

(B7) Show that for $d < -1$ there are exactly two functions $f$ from $R$ to $R$ such that, for all reals $x$ and $y$,

$$f(x + y) - f(x)f(y) = d \sin x \sin y.$$

(B8) For each of the conditions (a) and (b), find all functions $f$ from $R$ to $R$ such that the stated condition holds for all reals $x$, $y$.

(a) $f(x + f(x)f(y)) = f(x) + xf(y)$.

(b) $f(x + f(xy)) = f(x) + xf(y)$. 
(B9) Find all polynomials such that \( p(2) = 2 \) and
\[
p(x^2 - 1) = (p(x))^2 - 1
\]
for all reals \( x \).

(B10) Let \( f \) be a continuous function from \( R \) to \( R \) such that
\[
f(2x^2 - 1) = 2f(x)
\]
for all reals \( x \). Show that \( f(x) = 0 \) for all \( x \) between \(-1\) and \( 1\).

(B11) Let \( f \) and \( g \) be functions from \( R \) to \( R \) satisfying
\[
f(x + y) + f(x - y) = 2f(x)g(y)
\]
for all real \( x, y \). Prove that if \( f \) is not identically 0, and if \( |f(x)| \leq 1 \) for all \( x \), then \( |g(x)| \leq 1 \) for all \( x \).

Equations With Derivatives

(C1) Suppose \( f \) is a twice-differentiable function from \( R \) to \( R \), and that the following are true for all real \( x \).
Solve for \( f \).
\[
\begin{align*}
f(x) &= f(x + 2), \\
f'(x) &= f(x + 1) - 2.
\end{align*}
\]

(C2) Define polynomials \( f_n \) for \( n \geq 0 \) by \( f_0(x) = 1 \), \( f_n(0) = 0 \) for \( n \geq 1 \), and
\[
\frac{d}{dx}(f_{n+1}(x)) = (n + 1)f_n(x + 1)
\]
for \( n \geq 0 \). Find, with proof, the explicit factorization of \( f_{100}(1) \) into powers of distinct primes.

(C3) For each of the conditions (a) and (b), find all differentiable functions \( f \) from \( R \) to \( R \) such that that equation is true.
\[
\begin{align*}
(a) & \quad \frac{f(b) - f(a)}{b - a} = \frac{1}{2} (f'(a) + f'(b)). \\
(b) & \quad \frac{f(b) - f(a)}{b - a} = \sqrt{f'(a)f'(b)}.
\end{align*}
\]

(C4) Suppose \( f \) and \( g \) are nonconstant, differentiable functions from \( R \) to \( R \). Furthermore, suppose that for each pair of real numbers \( x \) and \( y \),
\[
\begin{align*}
f(x + y) &= f(x)f(y) - g(x)g(y), \\
g(x + y) &= f(x)g(y) + g(x)f(y).
\end{align*}
\]
If \( f'(0) = 0 \), prove that \( (f(x))^2 + (g(x))^2 = 1 \) for all \( x \).