(1) Find all nonnegative integer solutions to the following equations:
   (a) \(5^m + 3^m = 2^k\)
   (b) \(5^m + 3^n = 2^k\)

(2) Let \(p_k\) be the \(k\)th prime number, and suppose you have \(N\) sets of prime numbers, where each set \(S_i\) is of the form \(\{p_{i1}, p_{i2}, \ldots, p_{in_i}\}\) for some positive integer \(n_i\). Let \(T\) consist of all products of the form \(a_1a_2\cdots a_N\), where \(a_i \in S_i\) for all \(i\). When is \(T\) an interval of integers; i.e. when is \(T = \{n \in \mathbb{Z} : a \leq n \leq b\}\), for some \(a\) and \(b\)? (You may need to use Chebyshev’s Theorem: For all integers \(n \geq 2\), there is a prime number between \(n\) and \(2n\).)

(3) Prove that the polynomial \(P(x) = x^9 + 3x^8 - 5x^3 + 37\) cannot be factored into two polynomials which both have integer coefficients.

(4) Prove that if \(A\), \(B\), and \(C\) are real \(n \times n\) matrices, where \(n\) is not a multiple of 3, and

\[A^2 + B^2 + C^2 = AB + BC + CA,\]

then

\[\det((AB - BA) + (CA - AC) + (BC - CB)) = 0.\]
MAT 194/294/394/494, Fall 2006, Problem Set 1, Group 1

(1) The digits 1, 2, 3, 4, and 5 are each used once to compose a 5-digit number \( abcde \), such that the 3-digit number \( abc \) is divisible by 4, \( bcd \) is divisible by 5, and \( cde \) is divisible by 3. Find the 5-digit number \( abcde \).

(2) Find the equation(s) of the line(s) through the point \((2, 5)\) for which the \( y \)-intercept is a prime number and the \( x \)-intercept is an integer.

(3) A \( m \times n \) sheet of paper needs to be cut into \( mn \) 1 \( \times \) 1 squares, where \( m \) and \( n \) are positive integers. You may only cut one piece of paper at once; you cannot stack pieces to cut more than one at a time. What is the minimum number of cuts required? What is the maximum?

(4) Prove that for every natural \( n \), and for every real number \( x \neq k\pi/2 \) (\( t = 0, 1, \ldots, n; k \) any integer)

\[
\frac{1}{\sin(2x)} + \frac{1}{\sin(4x)} + \cdots + \frac{1}{\sin(2^n x)} = \cot x - \cot(2^n x).
\]