(A1) Find the volume of the region of points \((x, y, z)\) such that
\[
(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).
\]

(A2) Alice and Bob play a game in which they take turns removing stones from a heap that initially has \(n\) stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many \(n\) such that Bob has a winning strategy. (For example, if \(n = 17\), then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

(A3) Let \(1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots\) be a sequence defined by \(x_k = k\) for \(k = 1, 2, \ldots, 2006\), and \(x_{k+1} = x_k + x_{k-2005}\) for \(k \geq 2006\). Show that the sequence has 2005 consecutive terms each divisible by 2006.

(A4) Let \(S = \{1, 2, \ldots, n\}\) for some integer \(n > 1\). Say a permutation \(\pi\) of \(S\) has a local maximum at \(k \in S\) if

(i) \(\pi(k) > \pi(k + 1)\) for \(k = 1\);

(ii) \(\pi(k - 1) < \pi(k)\) and \(\pi(k) > \pi(k + 1)\) for \(1 < k < n\);

(iii) \(\pi(k - 1) < \pi(k)\) for \(k = n\).

(For example, if \(n = 5\) and \(\pi\) takes values at 1, 2, 3, 4, 5 of 2, 1, 4, 5, 3, then \(\pi\) has a local maximum of 2 at \(k = 1\), and a local maximum of 5 at \(k = 4\).) What is the average number of local maxima of a permutation of \(S\), averaging over all permutations of \(S\)?

(A5) Let \(n\) be a positive odd integer and let \(\theta\) be a real number such that \(\theta/\pi\) is irrational. Set \(a_k = \tan(\theta + k\pi/n)\), \(k = 1, 2, \ldots, n\). Prove that
\[
\sum_{k=1}^{n} a_k = \frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}
\]
is an integer, and determine its value.

(A6) Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.
(B1) Show that the curve \( x^3 + 3xy + y^3 = 1 \) contains only one set of three distinct points, \( A, B, \) and \( C, \) which are vertices of an equilateral triangle, and find its area.

(B2) Prove that, for every set \( X = \{x_1, x_2, \ldots, x_n\} \) of \( n \) real numbers, there exists a non-empty subset \( S \) of \( X \) and an integer \( m \) such that
\[
\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.
\]

(B3) Let \( S \) be a finite set of points in the plane. A linear partition of \( S \) is an unordered pair \( \{A, B\} \) of subsets of \( S \) such that \( A \cup B = S, A \cap B = \emptyset, \) and \( A \) and \( B \) lie on opposite sides of some straight line disjoint from \( S \) (\( A \) or \( B \) may be empty). Let \( L_S \) be the number of linear partitions of \( S \). For each positive integer \( n \), find the maximum of \( L_S \) over all sets \( S \) of \( n \) points.

(B4) Let \( Z \) denote the set of points in \( R^n \) whose coordinates are 0 or 1. (Thus \( Z \) has \( 2^n \) elements, which are the vertices of a unit hypercube in \( R^n \).) Given a vector subspace \( V \) of \( R^n \), let \( Z(V) \) denote the number of members of \( Z \) that lie in \( V \). Let \( k \) be given, \( 0 \leq k \leq n \). Find the maximum, over all vector subspaces \( V \subseteq R^n \) of dimension \( k \), of the number of points in \( V \cap Z \). [Editorial note: the proposers probably intended to write \( Z(V) \) for \( V \cap Z \), but this changes nothing.]

(B5) For each continuous function \( f : [0,1] \to R \), let \( I(f) = \int_0^1 x^2 f(x) \, dx \) and \( J(f) = \int_0^1 x (f(x))^2 \, dx \). Find the maximum value of \( I(f) - J(f) \) over all such functions \( f \).

(B6) Let \( k \) be an integer greater than 1. Suppose \( a_0 > 0, \) and define
\[
a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}
\]
for \( n \geq 0 \). Evaluate
\[
\lim_{n \to \infty} \frac{a_{n+1}^n}{n^k}.
\]