Power Series and Taylor Series

A power series is a series which looks like \( \sum_{k=0}^{\infty} a_k \cdot x^k \) or \( \sum_{k=0}^{\infty} a_k \cdot (x - a)^k \). Whether it converges can depend on the value of \( x \)! (Incidentally, \( 0^0 = 1 \) here.)

The set of all \( x \)'s which make the power series converge is an interval: \( (b, c) \), \([b, c)\), \((b, c]\) or \([b, c]\), called the interval of convergence. The quantity \( R = \frac{c - b}{2} \) is the radius of convergence. The center of the interval will be \( a \). (If you allow \( x \) to be a complex number, the “interval” becomes a circle.)
Example. Find the interval of convergence and the radius of convergence for the power series

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k} \cdot (x - 4)^k.$$

Use the Ratio Test!

$$L = \lim_{k \to \infty} \left| \frac{(-2)^{k+1}}{k + 1} \cdot (x - 4)^{k+1} \right| / \left| \frac{(-2)^k}{k} \cdot (x - 4)^k \right|$$

$$= \lim_{k \to \infty} \left( \frac{2^{k+1}}{2^k} \cdot \frac{k}{k + 1} \cdot \frac{|x - 4|^{k+1}}{|x - 4|^k} \right) = 2 \cdot 1 \cdot |x - 4|$$
We get convergence if $L < 1$, which happens when $|x - 4| < \frac{1}{2}$. We get divergence if $L > 1$, which is when $|x - 4| > \frac{1}{2}$. Thus $\frac{1}{2}$ is the radius of convergence.

If $|x - 4| = \frac{1}{2}$ (which is when $x = 4 \pm \frac{1}{2} = \frac{7}{2}$ or $\frac{9}{2}$), $L = 1$ and the Ratio Test is inconclusive; you have to try something else. In one case, you’ll use the Alternating Series test; the convergence test for the other series depends on what you get.
If $x = \frac{7}{2}$, then the power series becomes

$$\sum_{k=1}^{\infty} \frac{(-2)^k}{k} \cdot (x - 4)^k = \sum_{k=1}^{\infty} \frac{(-2)^k}{k} \cdot \left(\frac{7}{2} - 4\right)^k = \sum_{k=1}^{\infty} \frac{1}{k}$$

which is a divergent $p$-series.
If \( x = \frac{9}{2} \), then the power series becomes

\[
\sum_{k=1}^{\infty} \frac{(-2)^k}{k} \cdot \left( x - 4 \right)^k = \sum_{k=1}^{\infty} \frac{(-2)^k}{k} \cdot \left( \frac{9}{2} - 4 \right)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}
\]

which converges by the Alternating Series Test.

The interval of convergence is thus \( \left( \frac{7}{2}, \frac{9}{2} \right] \).
Exceptions for the interval of convergence:

- If \( L = 0 \) for all \( x \), the radius of convergence is \( \infty \) and the interval of convergence is \( (-\infty, +\infty) \).
- If \( L = \infty \) when \( x \neq a \), the radius of convergence is 0 and the interval of convergence is \( \{a\} \) (or \([a, a]\)).
If \( f(x) \) is a function which has infinitely many derivatives at \( x = a \), then the **Taylor series** for \( f(x) \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

where \( f^{(n)}(a) \) is the \( n \)th derivative of \( f(x) \), and

\[
 n! = 1 \cdot 2 \cdot \ldots \cdot n.
\]

\((0! = 1.)\)

The **MacLaurin series** for a function \( f(x) \) is the Taylor series at \( a = 0 \).
Example. Find the Taylor series for \( \ln(x) \) at \( x = 1 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \ln x )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{x} = x^{-1} )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( -x^{-2} )</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>( 2x^{-3} )</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>( -6x^{-4} )</td>
<td>-6</td>
</tr>
</tbody>
</table>

\[
0 + \frac{1}{1!}(x - 1)^1 + \frac{-1}{2!}(x - 1)^2 + \frac{2}{3!}(x - 1)^3 + \cdots
= (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - 1)^n
\]
Example. Find the MacLaurin series for \( \cos(x) \).

\[
\begin{array}{|c|c|c|}
\hline
n & f^{(n)}(x) & f^{(n)}(1) \\
\hline
0 & \cos x & 1 \\
1 & -\sin x & 0 \\
2 & -\cos x & -1 \\
3 & \sin x & 0 \\
4 & \cos x & 1 \\
5 & -\sin x & 0 \\
\hline
\end{array}
\]

\[
1 + \frac{0}{1!}x^1 + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}
\]
Some useful “basic” MacLaurin series:

\[ e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \]

\[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \]

\[ \ln(x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \]
Manipulation of Power Series: You can do any of the following to a power series for \( f(x) \) to get a power series for a related function:

- Replace \( x \) with \( Ax^n \) for some positive integer \( n \) (new power series is for \( f(Ax^n) \)).
- Multiply \( x^n \), for some integer \( n \) (new power series is for \( x^n f(x) \)).
- Differentiate term by term (new power series is for \( f'(x) \)).
- Integrate term by term (new power series is for \( \int f(x) \, dx \)).

The last three do not affect the Radius of Convergence.
Example.

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad (RC = 1)
\]

\[
\frac{1}{1 + x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots
\]

\[
= 1 - x^2 + x^4 - x^6 + \cdots \quad (RC = 1^*)
\]

\[
\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (RC = 1)
\]
* The Radius of Convergence for \( \frac{1}{1-x} \)'s power series is 1, so the series (converges) if \(|x|\) is (less than) 1.

Thus \( \frac{1}{1-(-x^2)} \)'s power series (converges) if \(|-x^2|\) is (less than) 1. Since \(|-x^2| > 1\) when \(|x|^2 > 1\) or \(|x| > 1\) (and the same for <), the RC of the new power series is 1 as well.
Note that convergence might be gained or lost at the endpoints of the Interval of Convergence, with the last three operations: The IC for the series for \( \frac{1}{1 + x^2} \) is \((-1, 1)\), but the IC for the series for \( \arctan(x) \) is \([-1, 1]\).
Series Convergence Tests

To recap: A sequence is a list of numbers: \( a_1, a_2, a_3, \ldots \) A series is when you try to add them up.

The \( n \)th partial sum is \( \sum_{k=1}^{n} a_k \).

\[
\sum_{k=1}^{\infty} a_k \text{ converges iff } \lim_{n \to +\infty} \sum_{k=1}^{n} a_k \text{ exists (is a real number).}
\]

Easier said than done, though.
Note that we can start the summation at any number and not affect convergence: \( \sum_{k=1}^{\infty} a_k \) converges iff \( \sum_{k=2}^{\infty} a_k \) converges, iff \( \sum_{k=3}^{\infty} a_k \) converges, iff \ldots They do converge to different things, though!

Geometric sequence: \( a_k = ar^k \) for some fixed \( a, r \).

The geometric series \( \sum_{k=0}^{\infty} ar^k \) converges (to \( \frac{a}{1-r} \)) if \( |r| < 1 \) and diverges if \( |r| \geq 1 \).
A geometric series and a Taylor series are the only ones where we will know what the series adds up to; otherwise, we will have to be content with knowing that a series adds up to something.
The idea of convergence tests is that if we know something about the **sequence**, we know something about the **series**.

- (Eventually) Positive Sequences: Integral test, Comparison Test, Limit Comparison Test.
- (Eventually) Alternating Sequences: Alternating Series Test
- Any Sequence: Ratio Test, Root Test
**The Test That Is Not To Be Named:** If $a_k$ does not converge to 0, then the series $\sum a_k$ diverges.

Example: $\sum_{k=4}^{\infty} \frac{2k}{2+k}$ diverges, since

$$\lim_{k \to \infty} \frac{2k}{2+k} = 2 \neq 0.$$ 

(If $a_k$ converges to 0, we don’t know what will happen; the series might converge or diverge.)
Integral Test: $a_k = f(k)$ for some function $f$ which is decreasing and which converges to 0. Then $\sum_{k=a}^{\infty} a_k$ converges iff $\int_{a}^{\infty} f(x) \, dx$ converges.

Example: $\sum_{k=3}^{\infty} \frac{1}{k + 1}$ diverges, because

$$\int_{3}^{\infty} \frac{1}{x + 1} \, dx = \lim_{b \to \infty} \left( \ln(b + 1) - \ln(3 + 1) \right) = \infty.$$
The \textbf{\textit{p-series}} are useful to know: \(\sum_{k=1}^{\infty} \frac{1}{k^p}\) converges if \(p > 1\) and diverges if \(p \leq 1\). This follows from the Integral Test.
Comparison Test: If \( 0 \leq a_k \) for all \( k \), and \( a_k \leq b_k \) (eventually), then you cannot have \( \sum a_k \) diverge and \( \sum b_k \) converge. Thus:

- If \( \sum a_k \) diverges, so does \( \sum b_k \).
- If \( \sum b_k \) converges, so does \( \sum a_k \).

Note that

- \( -1 \leq \sin(f(k)) \leq 1 \)
- \( -1 \leq \cos(f(k)) \leq 1 \)
- \( -1 \leq (-1)^k \leq 1 \)

for any function \( f \). These are useful estimates!
Example. Does \( \sum_{k=1}^{\infty} \frac{5 + \sin(k^2)}{k^2} \) converge?

Note that \(-1 \leq \sin(k^2) \leq 1\), so

\[
\frac{4}{k^2} \leq \frac{5 + \sin(k^2)}{k^2} \leq \frac{6}{k^2}.
\]

We recognize that \( \sum \frac{6}{k^2} \) converges, and it is bigger than \( \sum \frac{5 + \sin(k^2)}{k^2} \); hence \( \sum \frac{5 + \sin(k^2)}{k^2} \) also converges. (Here, \( a_k = \frac{5 + \sin(k^2)}{k^2} \) and \( b_k = \frac{6}{k^2} \).)
Limit Comparison Test: Idea: If $a_k \approx b_k$, then $\sum a_k$ and $\sum b_k$ both converge or both diverge.

More precisely: Suppose that $\lim_{k \to \infty} \frac{b_k}{a_k}$ is a positive real number. Then $\sum a_k$ and $\sum b_k$ both converge or both diverge.

You will be given an $a_k$; you need to find an appropriate $b_k$. This is usually done by removing lower powers in a polynomial, etc.
Example. Does \( \sum_{k=3}^{\infty} \frac{k - 5}{k^2 + 3k} \) converge?

Here, \( a_k = \frac{k - 5}{k^2 + 3k} \). When \( k \) is big, \( a_k \approx \frac{k}{k^2} = \frac{1}{k} \). Thus we should try \( b_k = \frac{1}{k} \).

Note that

\[
\lim_{k \to \infty} \frac{b_k}{a_k} = \lim_{k \to \infty} \frac{1}{k} \cdot \frac{k^2 + 3k}{k - 5} = \lim_{k \to \infty} \frac{k^2 + 3k}{k^2 - 5k} = 1,
\]

which is a positive real number. So \( \sum a_k \) does the same thing as \( \sum b_k \). Since \( \sum b_k \) diverges (it’s a \( p \)-series with \( p \leq 1 \)), so does \( \sum a_k \).
An Alternating Series is a series which alternates sign, term by term. Thus
\[ \frac{1}{3} - \frac{2}{7} + \frac{5}{13} - \frac{8}{21} + \cdots \]
is an alternating series;
\[ \frac{1}{3} - \frac{2}{7} - \frac{5}{13} + \frac{8}{21} - \cdots \]
is not an alternating series.

An alternating sequence frequently looks like
\[ (-1)^k \cdot f(k), \quad (-1)^{k\pm1} \cdot f(k), \quad \text{or} \quad \cos(k\pi) \cdot f(k), \]
where \( f(k) \) is a function that is always positive.
**Alternating Series Test:** If \( a_k \) is an alternating series, and \(|a_k|\) is decreasing, and \( \lim_{k \to \infty} a_k = 0 \), then \( \sum a_k \) converges.

Example: Does \( \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 + 1} \) converge?

This is an alternating series. \(|a_k| = \frac{1}{k^2 + 1}\), which converges to 0, and which is decreasing. Thus \( \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 + 1} \) converges.
We can also use the fact that, if $\sum a_k$ is an alternating series that converges, then we can approximate it:

$$\left| \sum_{k=a}^{\infty} a_k - \sum_{k=a}^{N} a_k \right| \leq |a_{N+1}|.$$

Since \[
\frac{1}{(1000)^2 + 1} \leq 10^{-6},
\]

$$\left| \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 + 1} - \sum_{k=2}^{999} \frac{(-1)^k}{k^2 + 1} \right| \leq \frac{1}{(1000)^2 + 1} \leq 10^{-6},$$

and $\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 + 1} = 0.575673547 \pm 10^{-6}$. 
**Ratio Test and Root Test:** For the Ratio Test, calculate
\[ L = \lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} \]; for the Root Test, calculate
\[ L = \lim_{k \to \infty} k^{\sqrt{|a_k|}}. \] Then

- If \( L < 1 \), the series \( \sum a_k \) converges (absolutely).
- If \( L > 1 \), the series \( \sum a_k \) diverges.
- If \( L = 1 \), the result is inconclusive: either convergence or divergence is possible.

The Ratio Test works best when you have factors which look like \( r^k \), \( k^r \), \( k! \), and constants. The Root Test should only be applied if \( a_k \) is something raised to the \( k \) power.
Example. Does \( \sum_{k=1}^{\infty} \frac{3^k}{k!} \) converge?

Let’s try the Ratio Test:

\[
L = \lim_{k \to \infty} \left( \frac{3^{k+1}}{(k+1)!} \cdot \frac{k!}{3^k} \right) = \lim_{k \to \infty} \frac{3}{k + 1} = 0.
\]

Since \( L < 1 \), the series converges (absolutely).
Example. Does \( \sum_{k=1}^{\infty} \frac{3^k}{k} \) converge?

Use the Ratio Test:

\[
L = \lim_{k \to \infty} \left( \frac{3^{k+1}}{k+1} \cdot \frac{k}{3^k} \right) = \lim_{k \to \infty} \frac{3k}{k+1} = 3.
\]

Since \( L > 3 \), the series diverges.

Don’t use the Ratio Test if \( a_k \) is a rational function! You will ALWAYS get \( L = 1 \) (no information)! (Do a Limit Comparison Test with a \( p \)-series instead.)
Example. Does $\sum_{k=5}^{\infty} \left(\frac{1}{3} + \frac{1}{2k}\right)^k$ converge?

Use the Root Test here:

$$L = \lim_{k \to \infty} k\sqrt[k]{\left(\frac{1}{3} + \frac{1}{2k}\right)^k} = \lim_{k \to \infty} \left(\frac{1}{3} + \frac{1}{2k}\right) = \frac{1}{3}.$$ 

Since $L < 1$, the series converges (absolutely).
Infinite series have the following property: If $\sum |a_k|$ converges, then so does $\sum a_k$. The reverse need not happen: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges, but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

If a series $\sum a_k$ converges, but $\sum |a_k|$ diverges, then the series $\sum a_k$ is said to converge conditionally; if $\sum |a_k|$ converges as well, then the series $\sum a_k$ converges absolutely.
If you want to check conditional convergence / absolute convergence, you need to test whether TWO series converge. For most of the problems you run across, checking whether $\sum a_k$ converges means you’ll be using the Alternating Series Test.
START

Does $\sum |a_k|$ converge?

$\rightarrow$ NO

Does $\sum a_k$ converge?

$\rightarrow$ NO

Divergence

$\rightarrow$ YES

Absolute Convergence

$\rightarrow$ YES

Conditional Convergence

$\rightarrow$ NO