(1) [30 points] For the matrix $A$ below, find a basis for the null space of $A$, the row space of $A$, the column space of $A$, the rank of $A$, and the nullity of $A$. The reduced row echelon form of $A$ is the matrix $R$ given below.

$$A = \begin{bmatrix} 1 & -2 & 4 & 3 & 6 \\ -1 & 2 & -4 & -3 & -6 \\ -2 & 4 & -8 & 3 & 15 \end{bmatrix} \quad R = \begin{bmatrix} 1 & -2 & 4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: The null space is the set of all $\vec{x}$ such that $A\vec{x} = \vec{0}$, or $R\vec{x} = \vec{0}$. The parameterization for these solutions is given by:

$$x_1 = 2r - 4s + 3t$$
$$x_2 = r$$
$$x_3 = s$$
$$x_4 = -3t$$
$$x_5 = t$$

so $x_3 = r \cdot 0 + s \cdot 1 + t \cdot -3$,

so a basis for the null space is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 3 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} t,$$

so a basis for the null space is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}.$$ Since there are three vectors in this set, the nullity is 3.

A basis for the column space consists of the columns of $A$ which have a pivot in them, namely

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\}.$$ A basis for the row space of $A$ is the nonzero rows of $R$, namely

$$\{[1, -2, 4, 0, -3], [0, 0, 0, 1, 3]\}.$$ Since there are two vectors in each of these sets, the rank of $A$ is 2.

Grading: +10 points for the null space basis, +5 points for the nullity, the row space basis, the column space basis, and the rank. Grading for common mistakes: −3 points for choosing the columns of $R$; −3 points for choosing rows from $A$. 

1
Let \( B = \begin{pmatrix} -1 & 3 & 1 \\ 0 & -2 & 1 \\ 4 & -3 & -2 \end{pmatrix} \) and \( C = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \\ 2 & -3 & -6 \end{pmatrix} \) be two ordered bases for \( \mathbb{R}^3 \).

(a) [10 points] Find the coordinates of the vector \( \vec{u} = \begin{pmatrix} -11 \\ 5 \\ 48 \end{pmatrix} \) with respect to the ordered basis \( B \).

\[
\begin{align*}
\begin{bmatrix} \vec{u} \end{bmatrix}_B &= \tilde{B}^{-1} \cdot \vec{u} \\
&= \begin{bmatrix} -1 & -3 & 1 \\ 0 & -1 & 1 \\ 4 & 9 & -2 \end{bmatrix}^{-1} \cdot \begin{pmatrix} -11 \\ 5 \\ 48 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 6 \\ 11 \end{pmatrix}.
\end{align*}
\]

Grading: +4 points for the basic formula \( \begin{bmatrix} \vec{u} \end{bmatrix}_B = \tilde{B}^{-1} \cdot \vec{u} \), +3 points for substitution, +3 points for calculation.

(b) [10 points] Find the change of basis matrix from \( C \) to \( B \).

\[
\begin{align*}
\tilde{B}^{-1} \cdot \tilde{C} &= \begin{bmatrix} -1 & -3 & 1 \\ 0 & -1 & 1 \\ 4 & 9 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \\ 2 & -3 & -6 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -7 & -17 \\ 0 & 3 & 8 \\ 3 & 1 & 5 \end{bmatrix}.
\end{align*}
\]

Grading: +4 points for the basic formula \( \tilde{B}^{-1} \cdot \tilde{C} \), +3 points for substitution, +3 points for calculation. Grading for common mistakes: +7 points (total) for \( \tilde{C}^{-1} \cdot \tilde{B} \); +3 points (total) for \( \tilde{B} \cdot \tilde{C}^{-1} \) or \( \tilde{C} \cdot \tilde{B}^{-1} \), +3 points (total) for including \( \vec{u} \) in the answer.
Let \( \vec{v}_1 = \begin{bmatrix} -1 \\ -3 \\ 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 9 \\ -9 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -3 \\ 3 \\ 3 \\ -3 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} -3 \\ -15 \\ -9 \\ -11 \end{bmatrix} \), and let \( W \) be the subspace spanned by these vectors.

(a) [15 points] The set \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} \) is linearly dependent. Find a linearly independent set of vectors whose span is \( W \), and the dimension of \( W \).

Solution: The first part is like finding the column space of a matrix: Choose the vectors which have pivots in their columns (in the RREF).

\[
\begin{bmatrix}
-1 & 3 & -3 & 11 \\
-3 & 9 & 3 & -3 \\
3 & -9 & 3 & -15 \\
1 & -3 & 3 & -11 \\
\end{bmatrix}
\begin{bmatrix}
1 & -3 & 0 & -2 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Thus \( \{\vec{v}_1, \vec{v}_3\} \) is one possible answer. Since there are two vectors in this set, the dimension of \( W \) is \( 2 \).

Grading: +5 points for the RREF, +5 points for choosing the vectors, +5 points for finding the dimension. Grading for common mistakes: +5 points (total) for finding a nontrivial linear combination of the vectors which adds up to \( \vec{0} \); +7 points (total) for finding the null space of the matrix above.

(b) [10 points] Is \( \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \) in \( W \)? Justify your answer.

Solution: A vector \( \vec{u} \) is in \( W \) if the system of linear equations \( [\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 | \vec{u}] \) has at least one solution. We can check for this by finding the RREF:

\[
\begin{bmatrix}
-1 & 3 & -3 & 11 & 2 \\
-3 & 9 & 3 & -3 & 0 \\
3 & -9 & 3 & -15 & 1 \\
1 & -3 & 3 & -11 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & -3 & 0 & -2 & 0 \\
0 & 0 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

This system has no solutions (see the third row), so the vector is not in \( W \).

Grading: +4 points for the initial augmented matrix, +3 points for the RREF, +3 points for the answer.
(4) [25 points] Let \( A \) be the matrix \[
\begin{bmatrix}
-1 & -2 & 4 \\
-1 & 0 & 2 \\
-1 & -1 & 3
\end{bmatrix}
\]. Find the eigenvalues of \( A \), and for ONE of the eigenvalues, find a basis for its eigenspace.

Solution: To find the eigenvalues, we need to determine when \( \det(A - \lambda I) = 0 \):

\[
0 = \det(A - \lambda I) = \begin{vmatrix}
-1 - \lambda & -2 & 4 \\
-1 & -\lambda & 2 \\
-1 & -1 & 3 - \lambda
\end{vmatrix}
\]

\[
= (-1 - \lambda)(-\lambda)(3 - \lambda) + (-2)(2)(-1) + 4(-1)(-1)
- (-1 - \lambda)(2)(-1) - (-\lambda)(4)(-1) - (3 - \lambda)(-1)(-2) = \cdots
\]

\[
= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda^2 - 2\lambda + 1) = -\lambda(\lambda - 1)^2,
\]

so the eigenvalues are \( 0 \) and \( 1 \).

To find the eigenvectors for \( \lambda = r \), find a basis for the null space of \( A - rI \).

For \( \lambda = 0 \),

\[
A - 0I = \begin{bmatrix}
-1 & -2 & 4 \\
-1 & 0 & 2 \\
-1 & -1 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

and the parameterization is \( x_1 = 2s, \ x_2 = s, \ x_3 = s \), so an eigenvector looks like \( s \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \) and a basis for the eigenspace is \( \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \).

For \( \lambda = 1 \),

\[
A - 1I = \begin{bmatrix}
-2 & -2 & 4 \\
-1 & -1 & 2 \\
-1 & -1 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and the parameterization is \( x_1 = -s + 2t, \ x_2 = s, \ x_3 = t \), so an eigenvector looks like \( s \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \) and a basis for the eigenspace is \( \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \).

Grading: +10 points for the eigenvalues, +15 points for the eigenspace basis. For the eigenvalues: +3 points for the \( \det(A - \lambda I) \) formula, +7 points for finding the roots. For the eigenspace: +4 points for \( A - \lambda I \), +4 points for the RREF, +4 points for the parameterization, +3 points for the basis. Grading for common mistakes: −3 points for an incomplete expansion by minors (for finding the determinant), −10 points if the RREF of \( A - \lambda I \) turned out to be \( I \) (this means you made a mistake somewhere).
(1) [30 points] For the matrix $A$ below, find a basis for the null space of $A$, the row space of $A$, the column space of $A$, the rank of $A$, and the nullity of $A$. The reduced row echelon form of $A$ is the matrix $R$ given below.

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -3 & 17 & 7 \\ 4 & -1 & 11 & 5 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Solution:** The null space is the set of all $\vec{x}$ such that $A\vec{x} = \vec{0}$, or $R\vec{x} = \vec{0}$. The parameterization for these solutions is given by:

$$x_1 = -2s - t$$
$$x_2 = 3s + t$$
$$x_3 = s$$
$$x_4 = t$$

so a basis for the null space is

$$\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ 

Since there are three vectors in this set, the nullity is $2$.

A basis for the column space consists of the columns of $A$ which have a pivot in them, namely

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix} \right\}.$$ 

A basis for the row space of $A$ is the nonzero rows of $R$, namely

$$\left\{ [1, 0, 2, 1], [0, 1, -3, -1] \right\}.$$ 

Since there are two vectors in each of these sets, the rank of $A$ is $2$.

Grading: +10 points for the null space basis, +5 points for the nullity, the row space basis, the column space basis, and the rank. Grading for common mistakes: −3 points for choosing the columns of $R$; −3 points for choosing rows from $A$. 

1
(2) Let $B = \begin{pmatrix}
-1 & -3 & -3 \\
4 & 13 & 11 \\
-3 & -8 & -11
\end{pmatrix}$ and $C = \begin{pmatrix}
1 & 2 & 4 \\
-2 & -5 & -12 \\
-4 & -11 & -27
\end{pmatrix}$ be two ordered bases for $\mathbb{R}^3$.

(a) [10 points] Find the coordinates of the vector $\vec{u} = \begin{pmatrix}
18 \\
-48 \\
-106
\end{pmatrix}$ with respect to the ordered basis $C$.

Solution: If $\tilde{C} = \begin{pmatrix}
1 & 2 & 4 \\
-2 & -5 & -12 \\
-4 & -11 & -27
\end{pmatrix}$, then the coordinates of $\vec{u}$ are

$$[\vec{u}]_C = \tilde{C}^{-1} \cdot \vec{u} = \begin{pmatrix}
1 & 2 & 4 \\
-2 & -5 & -12 \\
-4 & -11 & -27
\end{pmatrix}^{-1} \cdot \begin{pmatrix}
18 \\
-48 \\
-106
\end{pmatrix} = \begin{pmatrix}
2 \\
4 \\
2
\end{pmatrix}.$$

Grading: +4 points for the basic formula $[\vec{u}]_C = \tilde{C}^{-1} \cdot \vec{u}$, +3 points for substitution, +3 points for calculation.

(b) [10 points] Find the change of basis matrix from $B$ to $C$.

Solution: This matrix is

$$\tilde{C}^{-1} \cdot \tilde{B} = \begin{pmatrix}
1 & 2 & 4 \\
-2 & -5 & -12 \\
-4 & -11 & -27
\end{pmatrix}^{-1} \cdot \begin{pmatrix}
-1 & -3 & -3 \\
4 & 13 & 11 \\
-3 & -8 & -11
\end{pmatrix} = \begin{pmatrix}
-49 & -153 & -145 \\
50 & 157 & 147 \\
-13 & -41 & -38
\end{pmatrix}.$$

Grading: +4 points for the basic formula $\tilde{C}^{-1} \cdot \tilde{B}$, +3 points for substitution, +3 points for calculation. Grading for common mistakes: +7 points (total) for $\tilde{B}^{-1} \cdot \tilde{C}$; +3 points (total) for $\tilde{B} \cdot \tilde{C}^{-1}$ or $\tilde{C} \cdot \tilde{B}^{-1}$; +3 points (total) for including $\vec{u}$ in the answer.
(3) Let \( \vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} -5 & -2 \\ 0 & 2 \end{bmatrix}, \) and let \( W \) be the subspace spanned by these vectors.

(a) [15 points] The set \( \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \} \) is linearly dependent. Find a linearly independent set of vectors whose span is \( W \), and the dimension of \( W \).

Solution: The first part is like finding the column space of a matrix: Choose the vectors which have pivots in their columns (in the RREF).

Thus \( \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} \) is one possible answer. Since there are three vectors in this set, the dimension of \( W \) is \( 3 \).

Grading: +5 points for the RREF, +5 points for choosing the vectors, +5 points for finding the dimension. Grading for common mistakes: +5 points (total) for finding a nontrivial linear combination of the vectors which adds up to \( \vec{0} \); +7 points (total) for finding the null space of the matrix above.

(b) [10 points] Is \( \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix} \) in \( W \)? Justify your answer.

Solution: A vector \( \vec{u} \) is in \( W \) if the system of linear equations \( [\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 | \vec{u}] \) has at least one solution. We can check for this by finding the RREF:

This system has no solutions (see the fourth row), so the vector is not in \( W \).

Grading: +4 points for the initial augmented matrix, +3 points for the RREF, +3 points for the answer.
(4) [25 points] Let $A$ be the matrix $\begin{bmatrix} 5 & -2 & -3 \\ 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix}$. Find the eigenvalues of $A$, and for ONE of the eigenvalues, find a basis for its eigenspace.

**Solution:** To find the eigenvalues, we need to determine when $\det(A - \lambda I) = 0$:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 5 - \lambda & -2 & -3 \\ 1 & 2 - \lambda & -3 \\ 1 & -2 & 1 - \lambda \end{vmatrix} = (5 - \lambda)(2 - \lambda)(1 - \lambda) + (6) + (6) + 3(2 - \lambda) - 6(5 - \lambda) + 2(1 - \lambda) = \cdots = -\lambda^3 + 8\lambda^2 - 16\lambda = -\lambda(\lambda^2 - 8\lambda + 16) = -\lambda(\lambda - 4)^2,$$

so the eigenvalues are $0$ and $4$.

To find the eigenvectors for $\lambda = r$, find a basis for the null space of $A - rI$.

For $\lambda = 0$,

$$A - 0I = \begin{bmatrix} 5 & -2 & -3 \\ 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the parameterization is $x_1 = s$, $x_2 = s$, $x_3 = s$, so an eigenvector looks like $s \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and a basis for the eigenspace is $\begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$.

For $\lambda = 4$,

$$A - 4I = \begin{bmatrix} 1 & -2 & -3 \\ 1 & -2 & -3 \\ 1 & -2 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the parameterization is $x_1 = 2s + 3t$, $x_2 = s$, $x_3 = t$, so an eigenvector looks like $s \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and a basis for the eigenspace is $\begin{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$.

Grading: +10 points for the eigenvalues, +15 points for the eigenspace basis. For the eigenvalues: +3 points for the $\det(A - \lambda I)$ formula, +7 points for finding the roots. For the eigenspace: +4 points for $A - \lambda I$, +4 points for the RREF, +4 points for the parameterization, +3 points for the basis. Grading for common mistakes: −3 points for an incomplete expansion by minors (for finding the determinant), −10 points if the RREF of $A - \lambda I$ turned out to be $I$ (this means you made a mistake somewhere).