(1) (15 points) Find $\frac{dy}{dx}$ at (1, 0) if $x$ and $y$ are related by the following equation:

$$3x^2 + \tan y - e^{x+2y} = 3 - e.$$

\textbf{Solution: Use implicit differentiation:}

$$3x^2 + \tan y - e^{x+2y} = 3 - e$$

$$6x + \sec^2 y \cdot \frac{dy}{dx} - e^{x+2y} \cdot \left(1 + 2 \cdot \frac{dy}{dx}\right) = 0$$

$$\sec^2 y \cdot \frac{dy}{dx} - 2e^{x+2y} \cdot \frac{dy}{dx} = -6x + e^{x+2y}$$

$$(\sec^2 y - 2e^{x+2y}) \cdot \frac{dy}{dx} = -6x + e^{x+2y}$$

$$\frac{dy}{dx} = \frac{-6x + e^{x+2y}}{\sec^2 y - 2e^{x+2y}} = \frac{-6 + e}{1 - 2e}$$

\textbf{Grading:} +5 points for implicit differentiation, +5 points for solving for $\frac{dy}{dx}$, +5 points for substitution.
(2) Find the derivatives of the following functions.

(a) (10 points) $\sqrt{1 + 3^x \cdot \log_3 x}$

Solution: $\frac{1}{2} \cdot (1 + 3^x \cdot \log_3 x)^{-1/2} \cdot \left( e^x \cdot \frac{1}{x} \cdot \frac{1}{\ln 3} + \log_3 x \cdot 3^x \cdot \ln 3 \right)$.

Grading: +3 points for the chain rule, +3 points for the derivative of $1 + 3^x \cdot \log_3 x$, +2 + 2 points for the derivatives of $3^x$ and $\log_3 x$.

(b) (10 points) $\frac{3t^2 - 5}{2t^2 + t}$

Solution: $\frac{(2t^2 + t)(6t) - (3t^2 - 5)(4t + 1)}{(2t^2 + t)^2}$, which simplifies to $\frac{3t^2 + 20t + 5}{(2t^2 + t)^2}$ (although you didn’t need to simplify it).

Grading: +5 points for the quotient rule, +5 points for differentiating the numerator and denominator. Grading for common mistakes: +7 points (total) for the simplified answer with no supporting work; +5 points (total) for $\frac{6t}{4t + 1}$.

(c) (10 points) $x \tan^{-1}(x^2)$

Solution: $x \cdot \frac{1}{1 + (x^2)^2} \cdot 2x + \tan^{-1}(x^2)$.

Grading: +2 points for the product rule, +3 points for the derivative of $x$, +5 points for the derivative of $\tan^{-1}(x^2)$. Grading for common mistakes: +7 points (total) for writing $x \tan^{-1}(x^2)$ as $(x \tan^{-1}) \cdot (x^2)$.

(d) (10 points) $\tan(3 + 2 \cot x)$

Solution: $\sec^2(3 + 2 \cot x) \cdot 2 \cdot (-\csc^2 x)$.

Grading: +5 points for the chain rule, +5 points for differentiating the trigonometric functions.
(3) (15 points) Find an equation for the line tangent to the curve \( y = \cos^2 x \) at the point \( \left( \frac{\pi}{4}, \frac{1}{2} \right) \).

Solution: The slope of the tangent line is \( y' \) at \( x = \frac{\pi}{4} \), which is

\[
\frac{d}{dx} (\cos^2 x) \bigg|_{x=\pi/4} = (2 \cos x \cdot -\sin x) \bigg|_{x=\pi/4} = -1.
\]

An equation for the tangent line is

\[
y - \frac{1}{2} = -(x - \frac{\pi}{4}) \quad \text{which simplifies to} \quad y = -x + \left( \frac{1}{2} + \frac{\pi}{4} \right).
\]

Grading: +5 points for the derivative of \( \cos^2 x \), +5 points for substituting \( x = \frac{\pi}{4} \), +5 points for the equation of the line. Grading for common mistakes: +3 points of \( y' \) was calculated incorrectly; -3 points of the final slope was not a number (but involved \( x \)).

(4) (15 points) Find \( \lim_{x \to 0} x \cdot \csc(2x) \) algebraically.

Solution: Put this limit into a form that you can use l’Hospital’s Rule on:

\[
\lim_{x \to 0} x \cdot \csc(2x) = \lim_{x \to 0} \frac{x}{\sin(2x)} \overset{\text{L’Hôpital’s Rule}}{=} \lim_{x \to 0} \frac{1}{2 \cos(2x) \cdot 2} = \frac{1}{2}.
\]

Grading: +5 points for writing the product as a ratio, +5 points for l’Hospital’s Rule, +5 points for evaluating the new limit. Grading for common mistakes: -3 points for writing the limit as \( \lim_{x \to 0} \frac{x}{\cos(2x)} \); +7 points (total) for rewriting it as \( \lim_{x \to 0} \frac{\csc(2x)}{1/x} \); +7 points for taking logarithms first; +5 points (total) for using l’Hospital’s Rule right away.
(5) (15 points) The surface area of a rectangular box with length \( \ell \), width \( w \), and height \( h \) is

\[ A = 2\ell w + 2\ell h + 2wh. \]

The surface area of a certain box is increasing at the rate of 2 meters per minute, the width is decreasing at the rate of 1 meter per minute, and the height remains constant at 5 meters. At a particular moment, the width is 3.2 m, and the length is 15 m. Is the length increasing or decreasing at this moment, and at what rate?

Solution: You know that \( \frac{dA}{dt} = 2 \), \( \frac{dw}{dt} = -1 \), \( h = 5 \) (which is constant), \( w = 3.2 \), and \( \ell = 15 \). Take derivatives with respect to \( t \), then substitute this information:

\[ \frac{dA}{dt} = 2\ell \cdot \frac{dw}{dt} + 2w \cdot \frac{d\ell}{dt} + 2h \cdot \frac{d\ell}{dt} + 2h \cdot \frac{dw}{dt} \]

\[ 2 = 2 \cdot 15 \cdot (-1) + 2 \cdot 3.2 \cdot \frac{d\ell}{dt} + 2 \cdot 5 \cdot \frac{d\ell}{dt} + 2 \cdot 5 \cdot (-1) \]

\[ 42 = \frac{d\ell}{dt} \cdot 16.4 \]

so \( \frac{d\ell}{dt} = \frac{42}{16.4} \approx 2.561 \).

Grading: +5 points for taking derivatives with respect to \( t \), +5 points for substitution, +5 points for solving for \( \frac{d\ell}{dt} \). Grading for common mistakes: 2 points for \( \frac{dw}{dt} = 1 \); +7 points (total) for calculating the difference between \( \ell \) from one moment to the next; +5 points (total) for not taking derivatives.