MATH 270 Test #3N Solutions

(1) (15 points) A piece of wire 20 inches long is to be cut into two pieces. One piece will be bent into the shape of an equilateral triangle, and the other into a circle. Find out how to cut the wire to maximize the sum of the areas of the two regions. (The area of an equilateral triangle with side $s$ is $\frac{\sqrt{3}}{4}s^2$.)

Solution: Let $x$ be the length of the wire which is bent into a circle. Then since $x$ is the circumference of that circle, the radius is $\frac{x}{2\pi}$. The amount left over is $20-x$, which is bent into an equilateral triangle. Since this triangle has three sides, each has length $\frac{20-x}{3}$. The formula for the sum of the areas of these shapes is thus

$$A(x) = \pi \left( \frac{x}{2\pi} \right)^2 + \frac{\sqrt{3}}{4} \left( \frac{20-x}{3} \right)^2.$$

We want to find the maximum value of this function over the interval $[0, 20]$.

Now we look for critical points, places where the derivative is zero. ($A(x)$ is a polynomial, so the derivative exists everywhere.) Expanding $(20-x)^2$, we find out that

$$A(x) = \left( \frac{1}{4\pi} + \frac{\sqrt{3}}{36} \right) x^2 - \frac{\sqrt{3} \cdot 10}{9} x + \frac{100\sqrt{3}}{9},$$

so

$$A'(x) = 2 \left( \frac{1}{4\pi} + \frac{\sqrt{3}}{36} \right) x - \frac{\sqrt{3} \cdot 10}{9},$$

which has one critical point

$$x = \frac{\frac{\sqrt{3} \cdot 10}{9}}{2 \left( \frac{1}{4\pi} + \frac{\sqrt{3}}{36} \right)} \approx 7.5358.$$

The maximum occurs at one of the endpoints or at this value of $x$. Calculation shows that

$$A(0) \approx 19.245,$$

$$A(20) \approx 31.830,$$

$$A(7.5358) \approx 11.9937,$$

and the maximum occurs when $x = 20$; that is, when all of the wire is used to make the circle.

Grading: +4 points for the function $A(x)$, +3 points for finding the derivative, +4 points for finding the critical point, +4 points for evaluating $A(x)$ at the endpoints and the critical point. Grading for common mistakes: −3 points for not evaluating at the endpoints; −2 points for a function of two variables.
(2) (15 points) In the triangle $ABC$, the side $AB$ is increasing at a rate of 2 m/min, the side $AC$ is decreasing at a rate of 3 m/minute, and the angle $BAC$ is fixed at $3\pi/4$ radians. How fast is the distance $BC$ changing when $AB$ is 10 meters and $AC$ is 14 meters? Is the length of $BC$ increasing or decreasing? You may need to use the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos(\alpha).$$

Solution: Related to the equation above, $AB$ is $c$, $AC$ is $b$, angle $BAC$ is $\alpha$, and $BC$ is $a$. From the problem we know that $\frac{dc}{dt} = 2$, $\frac{db}{dt} = -3$ (since $b$ is DECREASING), and $\alpha = \frac{3\pi}{4}$ is constant. The goal is to find $\frac{da}{dt}$ when $c = 10$ and $b = 14$. Taking the derivative of the Law of Cosines with respect to $t$, we get the following equation:

$$2a \cdot \frac{da}{dt} = 2b \cdot \frac{db}{dt} + 2c \cdot \frac{dc}{dt} - 2 \left( b \cdot \frac{dc}{dt} + c \cdot \frac{db}{dt} \right) \cos(\alpha)$$

$$= 2(14)(-3) + 2(10)(2) - 2(14 \cdot 2 - 3 \cdot 10) \cos \left( \frac{3\pi}{4} \right)$$

$$\approx -46.82842712.$$  

To solve for $\frac{da}{dt}$, we need $a$, which we get from the original equation:

$$a = \sqrt{b^2 + c^2 - 2bc \cos(\alpha)} = \sqrt{(14)^2 + (10)^2 - 2(14)(10) \cos \left( \frac{3\pi}{4} \right)} \approx 22.22588353.$$

Finally,

$$\frac{da}{dt} \approx \frac{-46.82842712}{2a} \approx -1.053466,$$

with the units being meters per minute. The distance $a$ is thus decreasing.

Grading: +5 points for implicit differentiation, +5 points for finding $a$, +5 points for finding $\frac{da}{dt}$. Grading for common mistakes: −2 points for forgetting a minus sign on $\frac{d\alpha}{dt}$.
(3) (15 points) Find $\frac{dy}{dx}$ if $x^2y + x \cos y = 4$. (It is okay if your final answer involves $x$ and $y$.)

Solution: This is an implicit differentiation problem. Taking the derivative of the above equation, thinking of $y$ as a function of $x$, we get:

$$2xy + x^2 \cdot \frac{dy}{dx} + x(- \sin y) \frac{dy}{dx} + \cos y = 0$$

$$\frac{dy}{dx} \cdot (x^2 - x \sin y) = - \cos y - 2xy$$

$$\frac{dy}{dx} = \frac{- \cos y - 2xy}{x^2 - x \sin y}.$$ 

Grading: +8 points for implicit differentiation, +7 points for solving for $\frac{dy}{dx}$.

(4) Let $g(x) = (2x^2 - 16x + 34)e^x$.

(a) (10 points) Find the first and second derivatives of $g(x)$. Be sure to label which is which.

Solution:

$$g'(x) = (2x^2 - 16x + 34)e^x + e^x(4x - 16) = (2x^2 - 12x + 18)e^x.$$ 

To find the second derivative $g''(x)$, take the derivative of the derivative:

$$g'(x) = (2x^2 - 12x + 18)e^x + e^x(4x - 12) = (2x^2 - 8x + 6)e^x.$$ 

Grading: +5 points for each.

(b) (10 points) Find $\lim_{x \to +\infty} g(x)$ and $\lim_{x \to -\infty} g(x)$.

Solution: The limit

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} (2x^2 - 16x + 34)e^x = \lim_{x \to +\infty} (+\infty)(+\infty) = +\infty.$$ 

The other is more difficult; $g(x)$ has the indeterminate form $+\infty \cdot 0$ when $x \to -\infty$. It needs to be put into the form $\frac{\ast}{\ast}$, so that L’Hospital’s Rule can be used. Thus

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} (2x^2 - 16x + 34)e^x = \lim_{x \to -\infty} \frac{2x^2 - 16x + 34}{e^{-x}}\overset{L'H}{\to} \lim_{x \to -\infty} \frac{4x - 16}{(-1) \cdot e^{-x}}\overset{L'H}{\to} \lim_{x \to -\infty} \frac{4}{e^{-x}} = \frac{4}{+\infty} = 0.$$ 

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\(^{L'H}\) indicates where L’Hospital’s Rule has been used.)

Grading: +3 points for the first limit, +7 points for the second. Grading for common mistakes: −1 point for using L’Hospital’s Rule for the first limit; +3 points (total) for +\(\infty\) for the second limit; +4 points (total) for 0 for the second limit, with no work; +5 points (total) for “\(e^{-x}\) goes to 0 faster” for the second limit.

(c) (10 points) Sketch the graph of \(g(x)\). Indicate whether \(g(x)\) has any relative minimums, relative maximums, or inflection points.

Solution: To find all the information requested, we need to find out where \(g'(x)\) and \(g''(x)\) are zero. Factoring, we find that if

\[
0 = g'(x) = (3x^2 - 12x + 12)e^x = 3(x^2 - 4x + 4)e^x = 3(x - 2)^2e^x,
\]

then there is only one critical point, 2. Factoring \(g''(x)\), we find out that if

\[
0 = g''(x) = (3x^2 - 6x)e^x = 3x(x - 2)e^x,
\]

then there are two “critical points”, 0 and 2. By choosing suitable testing points, we find out that \(g'(x)\) is always increasing, so \(g(x)\) has no relative maximums or relative minimums, and \(g''(x)\) is negative (\(g(x)\) is concave down) when \(x\) is between 0 and 2 and is positive (\(g(x)\) is concave up) when \(x\) is less than 0 or greater than 2. The function \(g(x)\) has two inflection points, 0 and 2, and a rough sketch of \(g(x)\) is shown below.

Grading: +3 points for the relative maximum/relative minimum/inflection point information, +4 points for the graph, +3 points for supporting work (checking to see where \(g'(x)\) and \(g''(x)\) are positive and negative). Grading for common mistakes: +3 points (total) for the wrong graph and nothing else; +5 points (total) for the wrong graph and some information.
(5) (10 points) Find the linearization of \( f(x) = \tan(x) \) at \( x = \pi/4 \).

**Solution:** The function that the tangent line is a graph of is the linearization of the function \( f(x) \). It is

\[
L(x) = f(\pi/4) + f'(\pi/4)(x - \pi/4) = \tan(\pi/4) + \sec^2(\pi/4)(x - \pi/4) \\
= 1 + 2(x - \pi/4) = 2x + (1 - \pi/2).
\]

Grading: +3 points for the derivative of \( f(x) \), +3 points for finding the slope of the tangent line, +4 points for the linearization. Grading for common mistakes: −2 points for \( \sec^2(x - \pi/4) \).

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(6) (15 points) Find the derivative of \((x^2 + 1)^{\cos x}\).

**Solution:** Let \( y = (x^2 + 1)^{\cos x} \). Since this is a function of \( x \) raised to a power of a function of \( x \), neither of the formulas \( r x^{r-1} \) or \( a^x \ln a \) will work. The logarithm needs to be taken of both sides, and then implicit differentiation needs to be used:

\[
\ln y = \ln(x^2 + 1)^{\cos x} = \cos x \ln(x^2 + 1) \\
\frac{1}{y} \frac{dy}{dx} = \cos x \cdot \frac{1}{x^2 + 1} \cdot 2x + \ln(x^2 + 1) \cdot (-\sin x) \\
\frac{dy}{dx} = y \left[ \frac{2x \cos x}{x^2 + 1} - \sin x \ln(x^2 + 1) \right] \\
\frac{dy}{dx} = (x^2 + 1)^{\cos x} \cdot \left[ \frac{2x \cos x}{x^2 + 1} - \sin x \ln(x^2 + 1) \right] .
\]

Grading: +5 points for taking the logarithm of both sides, +5 points for implicit differentiation, +5 points for substituting for \( y \). Grading for common mistakes: +7 points (total) for no work but the correct answer; +5 points (total) for \( \cos x (x^2 + 1)^{\cos x-1}(2x) \); −2 points if the final answer had \( y \) in it.
(1) (15 points) A piece of wire 20 inches long is to be cut into two pieces. One piece will be bent into the shape of a square, and the other into a circle. Find out how to cut the wire to maximize the sum of the areas of the two regions.

Solution: Let \( x \) be the length of the wire which is bent into a square. Then since \( x \) is the perimeter of that square, the side is \( \frac{x}{4} \). The amount left over is \( 20-x \), which is bent into a circle. Since the circumference is \( 20-x \), the radius is \( \frac{20-x}{2\pi} \). The formula for the sum of the areas of these shapes is thus

\[
A(x) = \pi \left( \frac{20-x}{2\pi} \right)^2 + \left( \frac{x}{4} \right)^2.
\]

We want to find the maximum value of this function over the interval \([0, 20]\).

Now we look for critical points, places where the derivative is zero. (\( A(x) \) is a polynomial, so the derivative exists everywhere.) Expanding \((20-x)^2\), we find out that

\[
A(x) = \left( \frac{1}{16} + \frac{1}{4\pi} \right) x^2 - \frac{10}{\pi} x + \frac{400}{4\pi},
\]

so

\[
A'(x) = 2 \left( \frac{1}{16} + \frac{1}{4\pi} \right) x - \frac{10}{\pi},
\]

which has one critical point

\[
x = \frac{10}{\pi} \cdot \frac{2}{\left( \frac{1}{16} + \frac{1}{4\pi} \right)} \approx 11.209.
\]

The maximum occurs at one of the endpoints or at this value of \( x \). Calculation shows that

\[
A(0) \approx 33, \quad A(20) \approx 25, \quad A(11.209) \approx 14,
\]

and the maximum occurs when \( x = 0 \); that is, when all of the wire is used to make the circle.

Grading: +4 points for the function \( A(x) \), +3 points for finding the derivative, +4 points for finding the critical point, +4 points for evaluating \( A(x) \) at the endpoints and the critical point. Grading for common mistakes: −3 points for not evaluating at the endpoints; −2 points for a function of two variables.
(2) (15 points) In the triangle $ABC$, the side $AB$ is increasing at a rate of 2 m/min, the angle $BAC$ is decreasing at a rate of 0.1 radians per minute, and the side $AC$ is fixed at 14 m. How fast is the distance $BC$ changing when $AB$ is 10 meters and angle $BAC$ is $3\pi/4$ radians? Is the length of $BC$ increasing or decreasing? You may need to use the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos(\alpha).$$

**Solution:** Related to the equation above, $AB$ is $c$, angle $BAC$ is $\alpha$, $AC$ is $b$, and $BC$ is $a$. From the problem we know that $\frac{dc}{dt} = 2$, $\frac{d\alpha}{dt} = -0.1$ (since $\alpha$ is DECREASING), and $b = 14$ is constant. The goal is to find $\frac{da}{dt}$ when $c = 10$ and $\alpha = \frac{3\pi}{4}$. Taking the derivative of the Law of Cosines with respect to $t$, we get the following equation:

$$2a \cdot \frac{da}{dt} = 2c \cdot \frac{dc}{dt} - b \cdot \left( 2c(-\sin \alpha) \cdot \frac{d\alpha}{dt} + \cos \alpha \cdot 2 \cdot \frac{dc}{dt} \right)$$

$$= 2(10)(2) - 14 \left[ 2(10) \left( -\sin \frac{3\pi}{4} \right) \cdot (-0.1) + \cos \frac{3\pi}{4} \cdot 2 \cdot 2 \right]$$

$$\approx 59.79898987.$$

To solve for $\frac{da}{dt}$, we need $a$, which we get from the original equation:

$$a = \sqrt{b^2 + c^2 - 2bc \cos(\alpha)} = \sqrt{(14)^2 + (10)^2 - 2(14)(10) \cos \left( \frac{3\pi}{4} \right)} \approx 22.22588353.$$

Finally,

$$\frac{da}{dt} \approx \frac{59.79898987}{2a} \approx 1.3452556,$$

with the units being meters per minute. The distance $a$ is thus increasing.

Grading: +5 points for implicit differentiation, +5 points for finding $a$, +5 points for finding $\frac{da}{dt}$. Grading for common mistakes: −2 points for forgetting a minus sign on $\frac{d\alpha}{dt}$. 

2
(3) (15 points) Find $\frac{dy}{dx}$ if $xy - x^2 \sin y = 5$. (It is okay if your final answer involves $x$ and $y$.)

Solution: This is an implicit differentiation problem. Taking the derivative of the above equation, thinking of $y$ as a function of $x$, we get:

$$x \cdot \frac{dy}{dx} + y - \left(x^2 \cdot \cos y \cdot \frac{dy}{dx} + \sin y \cdot 2x\right) = 0$$

$$(x - x^2 \cdot \cos y) \frac{dy}{dx} = \sin y \cdot 2x - y$$

$$\frac{dy}{dx} = \frac{2x \sin y - y}{x - x^2 \cos y}.$$ 

Grading: +8 points for implicit differentiation, +7 points for solving for $\frac{dy}{dx}$.

(4) Let $g(x) = (3x^2 - 18x + 30)e^x$.

(a) (10 points) Find the first and second derivatives of $g(x)$. Be sure to label which is which.

Solution:

$$g'(x) = (3x^2 - 18x + 30)e^x + e^x(6x - 18) = (3x^2 - 12x + 12)e^x.$$ 

To find the second derivative $g''(x)$, take the derivative of the derivative:

$$g'(x) = (3x^2 - 12x + 12)e^x + e^x(6x - 12) = (3x^2 - 6x)e^x.$$ 

Grading: +5 points for each.

(b) (10 points) Find $\lim_{x \to +\infty} g(x)$ and $\lim_{x \to -\infty} g(x)$.

Solution: The limit

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} (3x^2 - 18x + 30)e^x = \lim_{x \to +\infty} (+\infty)(+\infty) = +\infty.$$ 

The other is more difficult; $g(x)$ has the indeterminate form $+\infty \cdot 0$ when $x \to -\infty$. It needs to be put into the form $\frac{\ast}{\ast}$ so that L’Hospital’s Rule can be used. Thus

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} (3x^2 - 18x + 30)e^x = \lim_{x \to -\infty} \frac{3x^2 - 18x + 30}{e^{-x}} \overset{L'H}{=} \lim_{x \to -\infty} \frac{6x - 18}{(-1)e^{-x}} \overset{L'H}{=} \lim_{x \to -\infty} \frac{6}{e^{-x}} = \frac{4}{+\infty} = 0.$$
\( L' \) indicates where L'Hospital’s Rule has been used.

Grading: +3 points for the first limit, +7 points for the second. Grading for common mistakes: −1 point for using L'Hospital’s Rule for the first limit; +3 points (total) for +\( \infty \) for the second limit; +4 points (total) for 0 for the second limit, with no work; +5 points (total) for “\( e^{-x} \) goes to 0 faster” for the second limit.

(c) (10 points) Sketch the graph of \( g(x) \). Indicate whether \( g(x) \) has any relative minimums, relative maximums, or inflection points.

Solution: To find all the information requested, we need to find out where \( g'(x) \) and \( g''(x) \) are zero. Factoring, we find that if

\[
0 = g'(x) = (2x^2 - 12x + 18)e^x = 2(x^2 - 6x + 9)e^x = 2(x - 3)^2e^x,
\]

then there is only one critical point, 3. Factoring \( g''(x) \), we find out that if

\[
0 = g''(x) = (2x^2 - 8x + 6)e^x = 2(x^2 - 4x + 3)e^x = 2(x - 1)(x - 3)e^x,
\]

then there are two “critical points”, 3 and 1. By choosing suitable testing points, we find out that \( g'(x) \) is always increasing, so \( g(x) \) has no relative maximums or relative minimums, and \( g''(x) \) is negative (\( g(x) \) is concave down) when \( x \) is between 1 and 3 and is positive (\( g(x) \) is concave up) when \( x \) is less than 1 or greater than 3. The function \( g(x) \) has two inflection points, 1 and 3, and a rough sketch of \( g(x) \) is shown below.

![Graph of g(x)](image)

Grading: +3 points for the relative maximum/relative minimum/inflection point information, +4 points for the graph, +3 points for supporting work (checking to see where \( g'(x) \) and \( g''(x) \) are positive and negative). Grading for common mistakes: +3 points (total) for the wrong graph and nothing else; +5 points (total) for the wrong graph and some information.
(5) (10 points) Find the linearization of \( f(x) = \cot(x) \) at \( x = \pi/4 \).

**Solution:** The function that the tangent line is a graph of is the linearization of the function \( f(x) \). It is

\[
L(x) = f(\pi/4) + f'(\pi/4)(x - \pi/4) = \cot(\pi/4) + \csc^2(\pi/4)(x - \pi/4)
\]

\[
= 1 - 2(x - \pi/4) = 2x + (1 + \pi/2).
\]

Grading: +3 points for the derivative of \( f(x) \), +3 points for finding the slope of the tangent line, +4 points for the linearization. Grading for common mistakes: −2 points for \( \csc^2(x - \pi/4) \).

(6) (15 points) Find the derivative of \( (x^2 - 1)^{\sin x} \).

**Solution:** Let \( y = (x^2 - 1)^{\sin x} \). Since this is a function of \( x \) raised to a power of a function of \( x \), neither of the formulas \( r x^{r-1} \) or \( a^x \ln a \) will work. The logarithm needs to be taken of both sides, and then implicit differentiation needs to be used:

\[
\ln y = \ln(x^2 - 1)^{\sin x} = \sin x \ln(x^2 - 1)
\]

\[
\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \cdot \frac{1}{x^2 - 1} \cdot 2x + \ln(x^2 - 1) \cdot \cos x
\]

\[
\frac{dy}{dx} = y \left[ \frac{2x \sin x}{x^2 - 1} + \cos x \ln(x^2 - 1) \right]
\]

\[
\frac{dy}{dx} = (x^2 - 1)^{\sin x} \cdot \left[ \frac{2x \sin x}{x^2 - 1} + \cos x \ln(x^2 - 1) \right].
\]

Grading: +5 points for taking the logarithm of both sides, +5 points for implicit differentiation, +5 points for substituting for \( y \). Grading for common mistakes: +7 points (total) for no work but the correct answer; +5 points (total) for \( \sin x(x^2 - 1)^{\sin x-1}(2x) \); −2 points if the final answer had \( y \) in it.