1. **[15 pts]** Find the area of the region between the lines $x = 0$ and $x = 10$, and between $y = 0$ and $y = f(x)$, where $f(x) = 2e^{0.3x}$. Give an exact answer: **You will lose points if you only give a decimal approximation.** It is okay to have $e$ in your final answer. (Note that $f(x) > 0$ for all relevant values of $x$.)

Solution: The area of this region is given by the definite integral $\int_0^{10} 2e^{0.3x} \, dx = 2 \int_0^{10} e^{0.3x} \, dx$. Since $e^{0.3x}$ is not one of the basic functions, you need to use substitution. Let $u = 0.3x$; then $\frac{du}{dx} = 0.3$, so $du = 0.3 \, dx$. Then

$$2 \int_0^{10} e^{0.3x} \, dx = 2 \int_0^* 2 \cdot 0.3 \, du = \left[ \frac{2}{0.3} \cdot e^u \right]^*_0 = \left[ \frac{2}{0.3} \cdot e^{0.3x} \right]_0^{10} = \frac{2}{0.3} e^3 - \frac{2}{0.3} \approx 127.2369128.$$  

Grading: +5 points for setting up the integral, +5 points for finding the antiderivative, +5 points for substituting 0 and 10 for $x$ in the antiderivative. Grading for common mistakes: -3 points for an approximation; +7 points (total) for approximation using a Riemann sum; +7 points (total) for the area of a square or a rectangle (this region is not a square or a triangle; one side isn’t straight).

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2. **[10 pts]** Approximate the following integral by finding the right hand sum, using $n = 5$.

$$\int_0^2 e^{-t^2} \, dt$$

Solution: There are three main steps for setting up a Riemann sum:

(a) Find $\Delta x$. $\Delta x = \frac{b - a}{n} = \frac{2 - 0}{5} = 0.4$.

(b) Find the intervals. Since $n$ isn’t too large here, the intervals can be written out explicitly: $[0, 0.4], [0.4, 0.8], [0.8, 1.2], [1.2, 1.6], [1.6, 2]$.

(c) Set up the Riemann sum. For the right hand sum, the function $f(x) = e^{-t^2}$ needs to be evaluated at the right-hand endpoints, then multiplied by $\Delta x$, and then added up, to get

$$f(0.4) \cdot \Delta x + f(0.8) \cdot \Delta x + f(1.2) \cdot \Delta x + f(1.6) \cdot \Delta x + f(2) \cdot \Delta x = e^{-0.4^2}(0.4) + e^{-0.8^2}(0.4) + e^{-1.2^2}(0.4) + e^{-1.6^2}(0.4) + e^{-2^2}(0.4) = 0.6847937404.$$  

Grading: +3 points for $\Delta x$, +3 points for the intervals (or the function $x_k$), +4 points for setting up the right hand sum. Grading for common mistakes: +7 points (total) for a correct answer with no work; +5 points (total) for $e^{-4} - e^0$. 

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1
3. [15 pts] The temperature in Phoenix, AZ, between 8:00 a.m. and noon on a certain day was given by the function \( T(x) = \frac{3}{8}x^2 - 3x + 66x^2 - \frac{1}{2}x + 75 \), where \( x \) is the number of hours since the previous midnight. (Thus \( x = 11 \) represents 11:00 a.m.) What was the average temperature between 8:00 and 9:00 a.m.?

Solution: To find the average value of a function, use the average value formula \( \frac{1}{b-a}\int_a^b f(x) \, dx \). Thus the average temperature (using this function) is

\[
\int_8^9 \frac{3}{8}x^2 - 3x + 66x^2 - \frac{1}{2}x + 75 \, dx.
\]

Because of a typo in the problem (the function was intended to be just \( \frac{3}{8}x^2 - 3x + 66 \)), full credit was given for setting up the problem.

Grading for common mistakes: +10 points (total) for the average rate of change \( \frac{f(b) - f(a)}{b - a} \).

4. The cost for purifying a certain number of gallons of water is given by \( C(x) = 100 + 0.68x^3 \), where \( x \) is in gallons, and \( C(x) \) is the cost in dollars. The purified water can be sold at a price of $40 per gallon. Your company is unionized, and part of the conditions is that the workers do not have to purify more than 4 gallons a week.

(a) [10 pts] How many gallons of water should your company purify every week to maximize your profit?

Solution: The goal is to maximize the profit, so the profit function needs to be put together. Profit is revenue minus cost, so

\[ P(x) = R(x) - C(x) = 40x - (100 + 0.68x^3). \]

The object is to maximize \( P(x) \), with \( x \) between 0 and 4. To do this, first look for critical points, where \( P'(x) = 40 - 0.68 \cdot 3x^2 = 0 \):

\[
40 - 0.68 \cdot 3x^2 = 0 \\
40 = 0.68 \cdot 3x^2 \\
\frac{40}{0.68 \cdot 3} = x^2, \quad \text{so} \quad x = \sqrt{\frac{40}{0.68 \cdot 3}} \approx 4.428.
\]

Since the only critical point is outside of the range, the maximum must occur at one of the endpoints. \( P(0) = -100 \) and \( P(4) = 16.48 \), so the company should purify 4 gallons of water a week, for a profit of $16.48.

Grading: +3 points for the profit function, +3 points for finding the critical point, +4 points for evaluating the profit function at the appropriate values, and the final answer. Grading for common mistakes: -2 points for saying the maximum was at 4.4.

(b) [5 pts] You will be meeting with the head of the union at your company. Are there any issues you should bring up, and if so, which ones?

Solution: The intended response is something along the lines of: According to the analysis above, the company can make a larger profit if the limitation of 4 gallons a week is raised to 4.5 gallons a week. +5 points were given for this answer, +3 points for something close to this.
5. Find all antiderivatives for each of the following functions.

(a) [10 pts] \((t + 1)(t^2 + 2t - 3)^{10}\)

Solution: Since this is not one of the basic functions, you will need to use substitution. Let 
\(u = t^2 + 2t - 3\), so that \(\frac{du}{dt} = 2t + 2\), and hence \(du = 2(t + 1)\,dt\). Then

\[
\int (t + 1)(t^2 + 2t - 3)^{10}\,dt = \int (t + 1)u^{10} \frac{1}{2(t + 1)}\,du = \int \frac{1}{2}u^{10}\,du = \frac{1}{2} \cdot \frac{u^{11}}{11} + C = \frac{(t^2 + 2t - 3)^{11}}{22} + C.
\]

Grading: +3 points for the substitution and finding \(du\) in terms of \(dt\), +3 points for doing the substitution, +4 points for finding the antiderivative and getting an answer in terms of \(t\). Grading for common mistakes: –1 point if the +C was forgotten; –2 points for a final answer in terms of \(u\).

(b) [10 pts] \(2^x + \sqrt[4]{x} - 3x^{-1}\)

Solution: This is the sum of three “basic functions”, so you can use the power rule and the rule for the antiderivatives of \(2^x\) and \(x^{-1}\):

\[
\int 2^x + x^{1/4} - 3x^{-1}\,dx = \frac{2^x}{\ln 2} + \frac{x^{5/4}}{5/4} - 3\ln x + C.
\]

Grading: +3 points for each antiderivative, +1 point for the +C.

(c) [10 pts] \(\frac{6x^2 - 2}{2x^3 - 2x + 1}\)

Solution: This is another problem where the function doesn’t have a quick rule. Let \(u = 2x^3 - 2x + 1\), so that \(\frac{du}{dx} = 6x^2 - 2\), or \(du = (6x^2 - 2)\,dx\). Then:

\[
\int \frac{6x^2 - 2}{2x^3 - 2x + 1}\,dx = \int \frac{1}{u}\,du = \ln u + C = \ln(2x^3 - 2x + 1) + C.
\]

Grading: Same as in part (a).
6. [15 pts] Sketch the graph of the function \( g(x) = x^4 - 2x^2 + 4 \). Find all relative maximums, relative minimums, and inflection points, if there are any.

Solution: To get this information, you need to find \( g'(x) \) and \( g''(x) \), and determine where \( g'(x) \) and \( g''(x) \) are positive and where they are negative. So:

\[
g'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1)
\]

\[
g''(x) = \frac{d}{dx}(4x^3 - 4x) = 12x^2 - 4 = 12(x^2 - 1/3),
\]

so \( g''(x) = 0 \) if \( x^2 - 1/3 = 0 \), or if \( x = \pm \sqrt{1/3} \). Then \( g'(x) \) is negative for values of \( x \) less than \(-1\), positive for values of \( x \) between \(-1 \) and \( 0 \), negative for values of \( x \) between \( 0 \) and \( 1 \), and positive for all values of \( x \) greater than \( 1 \). Hence the function \( g(x) \) is decreasing, increasing, decreasing, and increasing; hence it has two relative minimums (at \(-1 \) and \( 1 \)) and one relative maximum (at \( 0 \)).

Checking out \( g''(x) \), we see that \( g''(x) \) is positive for all \( x \) less than \(-\sqrt{1/3} \), negative for all \( x \) between \(-\sqrt{1/3} \) and \( \sqrt{1/3} \) and positive for all \( x \) greater than \( \sqrt{1/3} \). Since the second derivative changes sign at \(-\sqrt{1/3} \) and \( \sqrt{1/3} \), \( g(x) \) has two inflection points, at \(-\sqrt{1/3} \) and \( \sqrt{1/3} \).

This provides enough information to sketch a good graph of \( g(x) \):

Grading: +3 points for \( g'(x) \) and \( g''(x) \), +3 points for finding the critical points and possible inflection points, +3 points for supporting work for determining the relative mins/maxes and inflection points, +3 points for the graph, +3 points for giving the relative mins/maxes and inflection points. Grading for common mistakes: +10 points (total) for a graph but no information about \( g'(x) \) or \( g''(x) \).
1. **[15 pts]** Find the area of the region between the lines $x = 0$ and $x = 5$, and between $y = 0$ and $y = f(x)$, where $f(x) = 3e^{0.2x}$. Give an exact answer: You will lose points if you only give a decimal approximation. It is okay to have $e$ in your final answer. (Note that $f(x) > 0$ for all relevant values of $x$.)

**Solution:** The area of this region is given by the definite integral $\int_0^5 3e^{0.2x} \, dx = 3 \int_0^5 e^{0.2x} \, dx$. Since $e^{0.2x}$ is not one of the basic functions, you need to use substitution. Let $u = 0.2x$; then $\frac{du}{dx} = 0.2$, so $du = 0.2 \, dx$. Then

\[
2 \int_0^5 e^{0.2x} \, dx = 3 \int_u^1 e^u \cdot \frac{1}{0.2} \, du = \left[ \frac{3}{0.2} e^u \right]_0^1 = \left[ \frac{3}{0.2} e^{0.2x} \right]_0^5 = 3e^1 - 3e^0 \approx 25.774.
\]

Grading: +5 points for setting up the integral, +5 points for finding the antiderivative, +5 points for substituting 0 and 5 for $x$ in the antiderivative. Grading for common mistakes: −3 points for an approximation; +7 points (total) for approximation using a Riemann sum; +7 points (total) for the area of a square or a rectangle (this region is not a square or a triangle; one side isn’t straight).

2. **[10 pts]** Approximate the following integral by finding the left hand sum, using $n = 4$.

\[
\int_1^3 e^{t^2} \, dt
\]

**Solution:** There are three main steps for setting up a Riemann sum:

(a) Find $\Delta x$. $\Delta x = \frac{b - a}{n} = \frac{3 - 1}{4} = 0.5$.

(b) Find the intervals. Since $n$ isn’t too large here, the intervals can be written out explicitly: $[1,1.5]$, $[1.5,2]$, $[2,2.5]$, $[2.5,3]$.

(c) Set up the Riemann sum. For the left hand sum, the function $f(x) = e^{t^2}$ needs to be evaluated at the right-hand endpoints, then multiplied by $\Delta x$, and then added up, to get

\[
f(1) \cdot \Delta x + f(1.5) \cdot \Delta x + f(2) \cdot \Delta x + f(2.5) \cdot \Delta x
= e^{1^2}(0.5) + e^{1.5^2}(0.5) + e^{2^2}(0.5) + e^{2.5^2}(0.5) = 292.4084962.
\]

Grading: +3 points for $\Delta x$, +3 points for the intervals (or the function $x_k$), +4 points for setting up the left hand sum. Grading for common mistakes: +7 points (total) for a correct answer with no work; +5 points (total) for $e^9 - e^1$.  

1
3. [15 pts] The temperature in Tempe, AZ, between 8:00 a.m. and noon on a certain day was given by the function \( T(x) = \frac{5}{16}x^2 - \frac{1}{2}x + 75 \), where \( x \) is the number of hours since the previous midnight. (Thus \( x = 11 \) represents 11:00 a.m.) What was the average temperature between 9:00 and 10:00 a.m.?

Solution: To find the average value of a function, use the average value formula \( \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \). Thus the average temperature (using this function) is

\[
\int_{9}^{10} \frac{5}{16}x^2 - \frac{1}{2}x + 75 \, dx.
\]

Because of a typo in Form C, full credit was given for setting up the problem.

Grading for common mistakes: +10 points (total) for the average rate of change \( \frac{f(b) - f(a)}{b - a} \).

4. The cost for purifying a certain number of gallons of water is given by \( C(x) = 150 + 0.39x^3 \), where \( x \) is in gallons, and \( C(x) \) is the cost in dollars. The purified water can be sold at a price of $50 per gallon. Your company is unionized, and part of the conditions is that the workers do not have to purify more than 6 gallons a week.

(a) [10 pts] How many gallons of water should your company purify every week to maximize your profit?

Solution: The goal is to maximize the profit, so the profit function needs to be put together. Profit is revenue minus cost, so

\[
P(x) = R(x) - C(x) = 50x - (150 + 0.39x^3).
\]

The object is to maximize \( P(x) \), with \( x \) between 0 and 6. To do this, first look for critical points, where \( P'(x) = 50 - 0.39 \cdot 3x^2 = 0:\n\]

\[
50 - 0.39 \cdot 3x^2 = 0 \\
50 = 0.39 \cdot 3x^2 \\
\frac{50}{0.39 \cdot 3} = x^2, \quad \text{so} \quad x = \sqrt{\frac{50}{0.39 \cdot 3}} \approx 6.53776.
\]

Since the only critical point is outside of the range, the maximum must occur at one of the endpoints. \( P(0) = -150 \) and \( P(6) = 65.76 \), so the company should purify 6 gallons of water a week, for a profit of $65.76.

Grading: +3 points for the profit function, +3 points for finding the critical point, +4 points for evaluating the profit function at the appropriate values, and the final answer. Grading for common mistakes: −2 points for saying the maximum was at 6.53.

(b) [5 pts] You will be meeting with the head of the union at your company. Are there any issues you should bring up, and if so, which ones?

Solution: The intended response is something along the lines of: According to the analysis above, the company can make a larger profit if the limitation of 6 gallons a week is raised to 6.6 gallons a week. +5 points were given for this answer, +3 points for something close to this.
5. Find all antiderivatives for each of the following functions.

(a) [10 pts] \((t - 2)(t^2 - 4t + 1)^9\)

\[\text{Solution: Since this is not one of the basic functions, you will need to use substitution. Let } u = t^2 - 4t + 1, \text{ so that } \frac{du}{dt} = 2t - 4, \text{ and hence } du = 2(t - 2) \, dt. \text{ Then}\]
\[
\int (t - 2)(t^2 - 4t + 1)^9 \, dt = \int (t - 2)u^9 \cdot \frac{1}{2(t - 2)} \, du = \int \frac{1}{2} u^9 \, du = \frac{1}{2} \cdot \frac{u^{10}}{10} + C = \frac{(t^2 - 4t + 1)^{10}}{20} + C.
\]

Grading: +3 points for the substitution and finding \(du\) in terms of \(dt\), +3 points for doing the substitution, +4 points for finding the antiderivative and getting an answer in terms of \(t\). Grading for common mistakes: −1 point if the +\(C\) was forgotten; −2 points for a final answer in terms of \(u\).

(b) [10 pts] \(3^x - 2 \sqrt[3]{x} + x^{-1}\)

\[\text{Solution: This is the sum of three “basic functions”, so you can use the power rule and the rule for the antiderivatives of } 3^x \text{ and } x^{-1}:\]
\[
\int 3^x - 2x^{1/3} + x^{-1} \, dx = \frac{3^x}{\ln 3} - 2 \cdot \frac{x^{4/3}}{4/3} + \ln x + C.
\]

Grading: +3 points for each antiderivative, +1 point for the +\(C\).

(c) [10 pts] \(\frac{9x^2 - 10x}{3x^3 - 5x^2 + 3}\)

\[\text{Solution: This is another problem where the function doesn’t have a quick rule. Let } u = 3x^3 - 5x^2 + 3, \text{ so that } \frac{du}{dx} = 9x^2 - 10x, \text{ or } du = (9x^2 - 10x) \, dx. \text{ Then:}\]
\[
\int \frac{9x^2 - 10x}{3x^3 - 5x^2 + 3} \, dx = \int \frac{1}{u} \, du = \ln u + C = \ln(3x^3 - 5x^2 + 3) + C.
\]

Grading: Same as in part (a).
6. [15 pts] Sketch the graph of the function \( g(x) = x^4 - 8x^3 + 4 \). Find all relative maximums, relative minimums, and inflection points, if there are any.

Solution: To get this information, you need to find \( g'(x) \) and \( g''(x) \), and determine where \( g'(x) \) and \( g''(x) \) are positive and where they are negative. So:

\[
\begin{align*}
g'(x) &= 4x^3 - 24x^2 = 4x^2(x - 6) \\
g''(x) &= \frac{d}{dx}(4x^3 - 24x^2) = 12x^2 - 48x = 12x(x - 4),
\end{align*}
\]

Then \( g'(x) \) is negative for values of \( x \) less than 6 and positive for values of \( x \) greater than 6. Hence the function \( g(x) \) is decreasing, then increasing; hence it has a relative minimum at \( x = 6 \) and no relative maximums.

Checking out \( g''(x) \), we see that \( g''(x) \) is positive for all \( x \) less than 0, negative for all \( x \) between 0 and 4 and positive for all \( x \) greater than 4. Since the second derivative changes sign at 0 and 4, \( g(x) \) has two inflection points, at \( x = 0 \) and \( x = 4 \).

This provides enough information to sketch a good graph of \( g(x) \):

Grading: +3 points for \( g'(x) \) and \( g''(x) \), +3 points for finding the critical points and possible inflection points, +3 points for supporting work for determining the relative mins/maxes and inflection points, +3 points for the graph, +3 points for giving the relative mins/maxes and inflection points. Grading for common mistakes: +10 points (total) for a graph but no information about \( g'(x) \) or \( g''(x) \).