MATH 271 Test #4T Solutions

You do not need to evaluate the integrals in problems (2)--(5); just set them up.

(1) (20 points) Consider the curve which is represented by the parametric equations

\[ x = 1 + t + t^2, \]
\[ y = 1 + e^t, \]

where \( t \) is any real number. Find the equation of the line tangent to this curve at the point \((1, 2)\).

\textit{Solution:} The slope of the tangent line to a curve is \( \frac{dy}{dx} \). Here, \( x \) and \( y \) are given in terms of the parameter \( t \), so

\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t}{1 + 2t}. \]

To find the slope of the line tangent to the specific point \((1, 2)\), we need to find the value of \( t \) which gives \( x(t) = 1 \) and \( y(t) = 2 \). Hence, \( t \) must satisfy

\[ 1 + t + t^2 = 1 \quad \text{and} \quad 1 + e^t = 2. \]

Only one value of \( t \) will work, namely \( t = 0 \). The slope of the tangent line at the point \((1, 2)\) is \( \frac{e^0}{1 + 2(0)} = 1 \).

Now we have a point and the slope, so we can use the point-slope form of the equation of a line:

\[ y - 2 = 1(x - 1), \quad \text{or} \quad y = x + 1. \]

\textit{Grading:} +5 points for writing \( dy/dx \) in terms of \( t \), +5 points for finding \( t \), +5 points for the slope of the tangent line, +5 points for the equation of the tangent line.
(2) The polar coordinate equation \( r = 1 + \cos \theta \) traces out a cardioid, covering it exactly once, when \( \theta \) ranges from 0 to \( 2\pi \).

(a) (5 points) Sketch the graph of this cardioid.

Solution: 

\[ x = (1 + \cos \theta) \cos \theta = \cos \theta + \cos^2 \theta \quad \text{and} \]
\[ y = (1 + \cos \theta) \sin \theta = \sin \theta + 2 \sin \theta \cos \theta, \] so
\[ \frac{dx}{d\theta} = -\sin \theta + 2 \cos \theta (-\sin \theta) \quad \text{and} \]
\[ \frac{dy}{d\theta} = \cos \theta + \sin \theta (-\sin \theta) + \cos \theta \cos \theta. \]

You are told that when \( \theta \) ranges from 0 to \( 2\pi \), the cardioid is traced out exactly once, so \( a = 0 \) and \( b = 2\pi \). Putting all of this together, the perimeter of the cardioid is

\[ \int_{0}^{2\pi} \sqrt{(-\sin \theta + 2 \cos \theta(-\sin \theta))^2 + (\cos \theta + \sin \theta(-\sin \theta) + \cos \theta \cos \theta)^2} \, d\theta. \]

No further simplification is required.

Grading for common mistakes: -2 points for writing \((A + B)^2 \) as \(A^2 + B^2\); -3 points for finding the surface area (including a factor of \(2\pi y\) in the integrand); -3 points for using the formula \( \int \sqrt{\left( \int x \, dx \right)^2 + \left( \int y \, dy \right)^2} \, dt \) for arc length/perimeter; -5 points for the integral \( \int \sqrt{1 + (-\sin \theta)^2} \, d\theta \).
(3) (20 points) Set up an integral to find the surface area of the surface you get by rotating the curve \( y = 2 + \sqrt{4 - x^2} \) \((-2 \leq x \leq 2)\) around the \(x\)-axis.

**Solution:** The integral for the surface area of a curve rotated around the \(x\)-axis is
\[
\int_{a}^{b} 2\pi y \sqrt{1 + (y')^2} \, dx
\]
when the curve is given as a function of \(x\). Here, \(a\) and \(b\) are the upper and lower limits on \(x\), respectively, and
\[
y' = \frac{1}{2} (4 - x^2)^{-1/2} (-2x),
\]
so the surface area is
\[
\int_{-2}^{2} 2\pi (2 + \sqrt{4 - x^2}) \sqrt{1 + \left(\frac{1}{2} (4 - x^2)^{-1/2} (-2x)\right)^2} \, dx,
\]
which simplified is
\[
\int_{-2}^{2} 2\pi (2 + \sqrt{4 - x^2}) \sqrt{1 + \frac{x^2}{4 - x^2}} \, dx.
\]
(You did not need to simplify the integral, however.)

Grading for common mistakes: −3 points for the integral of \(\sqrt{1 + y^2} \, dx\).

(4) (20 points) Set up integrals to find the centroid of the region bounded by the curves \( y = \sqrt{3x} \) and \( y = x \).

**Solution:** To find the centroid of this region, you need to find three integrals: the area \(A = \int y_T - y_B \, dx\), the \(x\)-coordinate of the centroid, which is \(\bar{x} = \frac{1}{A} \int x(y_T - y_B) \, dx\), and the \(y\)-coordinate of the centroid, which is \(\bar{y} = \frac{1}{A} \int \frac{1}{2} (y_T^2 - y_B^2) \, dx\), where \(y_T\) is the curve on top, and \(y_B\) is the curve on the bottom.

⇒
To determine which function is the top and which one is the bottom, and what the limits on x are, the region needs to be found. It turns out that $\sqrt{3}x = x$ (i.e., the curves cross) at $x = 0$ and $x = 3$, and $\sqrt{3}x > x$ for all values of $x$ in between. Hence $y_T = \sqrt{3}x$ and $y_B = x$. Thus

$$A = \int y_T - y_B \, dx = \int_0^3 (\sqrt{3}x - x) \, dx$$

$$\bar{x} = \frac{1}{A} \int x(y_T - y_B) \, dx = \frac{1}{A} \int_0^3 x(\sqrt{3}x - x) \, dx$$

$$\bar{y} = \frac{1}{A} \int \frac{1}{2} (y_T^2 - y_B^2) \, dx = \frac{1}{A} \int_0^3 \frac{1}{2} (\sqrt{3}x)^2 - \frac{1}{2} x^2 \, dx.$$

Grading: +6 points for the area formula and integral, +7 points for the formula for $\bar{x}$; +7 points for $\bar{y}$. Grading for common mistakes: −5 points for not including limits; −2 points for $\bar{y} = \int \frac{1}{2} (y_T - y_B)^2 \, dx$.

(5) (20 points) Set up an integral to find the area of the region which is under the curve parameterized by the equations

\[ x = 1 + s^3 \]
\[ y = 4 + 3s^2 - s^4 \]

(where $s$ is a real number) and above the $x$-axis. Note that $y = 0$ only when $s = \pm 2$.

Solution: Setting up the integral for area when the region is described by parametric equations can be tricky; the hint was intended to let you know that nothing weird was going on. For instance, the curve is above the $x$ axis when $s$ is between $-2$ and $2$. (Also, the function for $x$ is strictly increasing, so the curve never crosses itself.) This lets you know that you can use the formula

$$\int_{x \text{ at the left}}^{x \text{ at the right}} y \, dx.$$

Since $x$ and $y$ are given in terms of $s$, the area is

$$\int_{s:x \text{ at the left}}^{s:x \text{ at the right}} y \cdot \frac{dx}{dt} \, dt = \int_{s=-2}^{s=2} (4 + 3s^2 - s^4)(3s^2) \, ds.$$

Grading: +5 points (total) were given for no integral; +15 points (total) were given for an integral with no limits, or the surface area of the region obtained by revolving the curve around the $x$-axis; −2 points if limits for $x$ were given

$$(\int_{-7}^9 \cdots \, dx); -1 \text{ point was given for the limits } \int_{0}^{2} \cdots \, ds.$$
(6) (Extra credit: 5 points each) Evaluate the following integrals:

(a) \[ \int x^2 \sin(2x) \, dx \]

Solution: Since the integrand is the product of two functions, integration by parts is worth a try. When doing integration by parts, an easier integral is desired, so we let \( u = x^2 \) and \( v' = \sin(2x) \), so that \( u' = 2x \) and \( v = -\frac{\cos(2x)}{2} \), and hence

\[
\int x^2 \sin(2x) \, dx = -\frac{x^2 \cos(2x)}{2} - \int 2x \left( -\frac{\cos(2x)}{2} \right) \, dx = -\frac{x^2 \cos(2x)}{2} + \int x \cos(2x) \, dx.
\]

The power of \( x \) has gone down, so this integral is simpler than the previous one. Now we do integration by parts again, with \( u = x \) and \( v' = \cos(2x) \), so that \( u' = 1 \) and \( v = \frac{\sin(2x)}{2} \), so that

\[
-\frac{x^2 \cos(2x)}{2} + \int x \cos(2x) \, dx = -\frac{x^2 \cos(2x)}{2} + \frac{x \sin(2x)}{2} - \int 1 \cdot \frac{\sin(2x)}{2} \, dx.
\]

This integral can be evaluated using the substitution \( y = 2x \). This gives a final answer of

\[ -\frac{x^2 \cos(2x)}{2} + \frac{x \sin(2x)}{2} + \frac{\cos(2x)}{4} + C. \]

(b) \[ \int \frac{1 + 4t}{\sqrt{1 + t + 2t^2}} \, dt \]

Solution: The radical with a quadratic function underneath it suggests a trig substitution in the most general case. It also indicates that the substitution \( u = 1 + t + 2t^2 \) should be tried (which is a lot easier). If \( u = 1 + t + 2t^2 \), then \( du/dt = 1 + 4t \), so \( dt = \frac{du}{1 + 4t} \). Then

\[
\int \frac{1 + 4t}{\sqrt{1 + t + 2t^2}} \, dt = \int \frac{1 + 4t}{\sqrt{u}} \cdot \frac{du}{1 + 4t} = \int \frac{1}{\sqrt{u}} \, du = \int u^{-1/2} \, du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{u} + C = 2\sqrt{1 + t + 2t^2} + C.
\]