(1) Rewrite the integral \( \int_{0}^{1} x^2 \sin^{-1} x \, dx \) in terms of another integral using integration by parts. You do not need to evaluate this second integral.

**Solution:** Use integration by parts, with \( u = \sin^{-1} x \) and \( v' = x^2 \). Then \( u' = \frac{1}{\sqrt{1-x^2}} \) and \( v = \frac{x^3}{3} \) and

\[
\int_{0}^{1} x^2 \sin^{-1} x \, dx = \left. \frac{x^3}{3} \cdot \sin^{-1} x \right|_{0}^{1} - \int_{0}^{1} \frac{x^3}{3} \cdot \frac{1}{\sqrt{1-x^2}} \, dx
\]

(2) Perform a trigonometric substitution on the integral \( \int \frac{y}{y^2 - \sqrt{y^2 - 16}} \, dy \). You do not need to evaluate the new integral.

**Solution:** The trigonometric substitution you need to use is \( y = 4 \sec \theta \). Then \( dy = 4 \sec \theta \tan \theta \, d\theta \) and \( \sqrt{y^2 - 16} = 4 \tan \theta \), and

\[
\int \frac{y}{y^2 - \sqrt{y^2 - 16}} \, dy = \int \frac{4 \sec \theta}{16 \sec^2 \theta - 4 \tan \theta} \cdot 4 \sec \theta \tan \theta \, d\theta = \int \frac{16 \sec^2 \theta \tan \theta}{16 \sec^2 \theta - 4 \tan \theta} \, d\theta
\]
(3) Find the form of the Partial Fraction Decomposition of \(\frac{2x^3 + 5x - 7}{x(x - 4)^2(x^2 - 2x + 7)^2}\). You may keep constants such as \(A, B, C, \ldots\) in your answer. (Note that \(x^2 - 2x + 7\) has no real roots.)

Solution: \[
\frac{A}{x} + \frac{B}{x - 4} + \frac{C}{(x - 4)^2} + \frac{Dx + E}{x^2 - 2x + 7} + \frac{Fx + G}{(x^2 - 2x + 7)^2}
\]

(4) Rewrite the integral \(\int \sin^2 \theta \cos^3 \theta \, d\theta\) as an integral of a polynomial. You do not need to evaluate this new integral.

Solution: The power of \(\cos \theta\) is odd, so we will want to make the substitution \(u = \sin \theta\) to turn the integral into the integral of a polynomial. The other factor of \(\cos^3 \theta\), \(\cos^2 \theta\), needs to be rewritten in terms of \(\sin \theta\).

\[
\int \sin^2 \theta \cos^3 \theta \, d\theta = \int \sin^2 \theta (1 - \sin^2 \theta) \cos \theta \, d\theta = \int u^2(1 - u^2) \, du
\]

The last integral comes from the substitution \(u = \sin \theta\), where \(du = \cos \theta \, d\theta\).

(5) Make an appropriate substitution and evaluate the following integral completely.
\(\int 3x^4 \tan(x^5) \, dx\)

Solution: Let \(u = x^5\); then \(du = 5x^4 \, dx\), and

\[
\int 3x^4 \tan(x^5) \, dx = \int 3x^4 \tan(u) \cdot \frac{du}{5x^4} = -\frac{3}{5} \int \tan u \, du = -\frac{3}{5} \ln |\cos u| + C
\]

\[
= -\frac{3}{5} \ln |\cos(x^5)| + C
\]
(6) Evaluate the integral $\int \cot(4\theta) \, d\theta$ completely.

**Solution:**

$$
\int \cot(4\theta) \, d\theta = \int \frac{\cos(4\theta)}{\sin(4\theta)} \, d\theta = \int \frac{du}{u} = \frac{1}{4} \int u \, du = \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln |\sin(4\theta)| + C
$$

(7) Evaluate $\int \tan^{-1} x \, dx$.

**Solution:** Use integration by parts, with $u = \tan^{-1} x$ and $v' = 1$, so that $u' = \frac{1}{1+x^2}$ and $v = x$, then substitute $w = 1 + x^2$:

$$
\int \tan^{-1} x \, dx = x \tan^{-1} u - \int \frac{x}{1+x^2} \, dx = x \tan^{-1} x - \int \frac{x}{w} \cdot \frac{dw}{2x} \quad [dw = 2x \, dx]
$$

$$
= x \tan^{-1} x - \frac{1}{2} \int \frac{dw}{w} = x \tan^{-1} x - \ln w + C
$$

$$
= x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C
$$

(8) Evaluate $\int z^{\frac{3}{2}} + 2 \, dz$.

**Solution:** This can be done with integration by parts ($u = z$ and $v' = \frac{3}{2}z + 2$), or by substituting $u = z + 2$ ($du = dz$):

$$
\int z^{\frac{3}{2}} + 2 \, dz = \int (u - 2)^{\frac{3}{2}} \, du = \int u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \, du = \frac{3}{7} u^{\frac{7}{3}} - \frac{3}{4} u^{\frac{4}{3}} + C
$$

$$
= \frac{3}{7} (z + 2)^{\frac{7}{3}} - \frac{3}{2} (z + 2)^{\frac{4}{3}} + C
$$
(9) Evaluate $\int te^{2t} \, dt$, using integration by parts.

Solution: Let $u = t$ and $v' = e^{2t}$. (Choosing them the other way around results in an integral with a higher power of $t$ in it.) Then $u' = 1$ and $v = \frac{1}{2} e^{2t}$ and

$$\int te^{2t} \, dt = \frac{1}{2} e^{2t} \cdot t - \int 1 \cdot \frac{1}{2} e^{2t} \, dt = \frac{te^{2t}}{2} - \frac{e^{2t}}{4} + C$$

(10) Evaluate $\int \frac{2}{y^2 + 4} \, dy$.

Solution: We need to get the denominator into the form $(something)^2 + 1$, so that we can use the inverse tangent formula. So

$$\int \frac{2}{y^2 + 4} \, dy = \int \frac{2}{4 \left( \frac{y^2}{4} + 1 \right)} \, dy = \frac{1}{2} \int \frac{1}{\left( \frac{y}{2} \right)^2 + 1} \, dy$$

$$= \frac{1}{2} \int \frac{1}{u^2 + 1} \cdot 2 \, du \left[ u = y/2 \right]$$

$$= \int \frac{du}{u^2 + 1} = \tan^{-1}(u) + C = \tan^{-1} \left( \frac{y}{2} \right) + C$$

(11) Evaluate $\int \sec(\cos \phi) \tan(\cos \phi) \sin \phi \, d\phi$, by using an appropriate substitution.

Solution: Since $\cos \phi$ occurs twice in the integrand, a good choice for the substitution would be $u = \cos \phi$, with $du = -\sin \phi \, d\phi$, and

$$\int \sec(\cos \phi) \tan(\cos \phi) \sin \phi \, d\phi = - \int \sec u \tan u \, du = - \sec u + C = -\sec(\cos \phi) + C$$

(12) Evaluate $\int \sqrt{y^2 - 9} \, dy$.

Solution: This one was more difficult than intended. It can be done by using the trig substitution $y = 3 \sec \theta$, and then evaluating the resulting trigonometric integral $\int 9 \tan^2 \theta \sec \theta \, d\theta$. This integral is too difficult for a test problem, so everyone was given a free point for this problem.
(13) Evaluate \( \int \frac{x + 1}{x^2 + 2x + 3} \, dx \), by using an appropriate substitution.

**Solution:** Using the hint, there are two basic options: Let \( u = x + 1 \) or let \( u = x^2 + 2x + 3 \). The first doesn’t lead anywhere, so, if \( u = x^2 + 2x + 3 \), then \( du = (2x + 2) \, dx \), and

\[
\int \frac{x + 1}{x^2 + 2x + 3} \, dx = \int \frac{1}{u} \cdot \frac{du}{2x + 2} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln u + C = \frac{1}{2} \ln(x^2 + 2x + 3) + C
\]

(14) Evaluate \( \int \frac{1}{\sqrt{x - 1}} \, dx \).

**Solution:** \( \int \frac{1}{\sqrt{x - 1}} \, dx = \int (x - 1)^{-1/2} \, dx = 2(x - 1)^{1/2} + C \). The last equality comes from the substitution \( u = x - 1 \).

(15) Evaluate \( \int 3^t \, dt \).

**Solution:** If you don’t know this one right away, then work it out:

\[
\int 3^t \, dt = \int e^{t \ln 3} \, dt = \frac{e^{t \ln 3}}{\ln 3} + C = \frac{1}{\ln 3} \cdot 3^t + C
\]