Solutions to MAT 270 Test #3

Because there are two versions of the test, solutions will only be given for Form N (N). Differences from the Form S (S) version will be given. (The values for Form N appear above those of Form S in the curly braces {}.)

N  S (1) (15 points) Consider the curve \( \{ x^2y + x - y^2 + 1 = 0, xy^2 - y + x^3 - 3 = 0 \} \). Find the slope of the tangent line at the point (1, 2).

Solution: Because we do not have \( y \) explicitly as a function of \( x \), we need to use implicit differentiation. Remember that when using implicit differentiation, that the derivative of \( y^2 \) is \( 2y(y') \), etc. For N, we get the equation

\[
x^2(y') + y \cdot 2x + 1 - 2y(y') = 0,
\]

and when we solve for \( y' \), we get

\[
(x^2 - 2y)(y') = -2xy - 1, \quad \text{or} \quad y' = \frac{-2xy - 1}{x^2 - 2y}.
\]

This is the derivative at an arbitrary point. To find the slope of the tangent line at the point (1, 2), we put in \( x = 1 \) and \( y = 2 \), so that \( y' = \frac{-2(1)(2) - 1}{(1)^2 - 2(2)} = \frac{5}{3} \).

When we use implicit differentiation for S, we get

\[
x \cdot 2y(y') + y^2 - (y') + 3x^2 = 0
\]

\[
(2xy - 1)(y') = -y^2 - 3x^2
\]

\[
y' = \frac{-y^2 - 3x^2}{2xy - 1},
\]

and when we substitute \( x = 1 \) and \( y = 2 \), we find that the slope of the line is \( \frac{-(2)^2 - 3(1)^2}{2(1)(2) - 1} = \frac{-7}{3} \).

Grading: This problem was graded on a 0–5–10–15-point basis. Grading for common mistakes: -3 points for not putting in \( x = 1 \) and \( y = 2 \); -3 points for solving (incorrectly) for \( y \) first.
(2) (15 points) Find the largest and smallest values that \( \left\{ \frac{x^3 - 5x^2 + 7x - 3}{x^3 - 3x^2 + 4} \right\} \) attains on the interval \( \left\{ \begin{align*} {[0, 3]} \\ {[-2, 2]} \end{align*} \right\} \), and the value(s) of \( x \) where they occur.

**Solution:** The largest and smallest values can only occur at an endpoint or at a critical point, so we need to find these first. For \( N \), we find the critical points by looking at the derivative of \( f(x) = x^3 - 5x^2 + 7x - 3 \):

\[
f'(x) = 3x^2 - 10x + 7 = (3x - 7)(x - 1).
\]

Since \( f'(x) \) is defined for all values of \( x \), the critical points are those values of \( x \) where \( f'(x) = 0 \). Looking at the factored form, we see that the critical points are \( x = 1 \) and \( 7/3 \). We also need to consider the endpoints, because a minimum or maximum can also occur there. So, we have the four candidates 0, 1, 7/3, and 3. Now we evaluate \( f(x) \) at each one of these points to find out where the largest and smallest values occur:

\[
\begin{align*}
f(0) &= -3 \\
f(1) &= 1 - 5 + 7 - 3 = 0 \\
f(7/3) &= (7/3)^3 - 5(7/3)^2 + 7(7/3) - 3 \approx -1.185 \ldots \\
f(3) &= 27 - 45 + 21 - 3 = 0
\end{align*}
\]

The minimum value that \( f(x) \) attains is \( -3 \), which occurs at \( x = 0 \); the largest value that \( f(x) \) attains is 0, when \( x = 1 \) or \( x = 3 \).

For \( S \), the procedure is the same; the numbers are different. Here we let \( g(x) = x^3 - 3x^2 + 4 \). Then

\[
g'(x) = 3x^2 - 6x = 3x(x - 2),
\]

so the critical points of \( g(x) \) are \( x = 0 \) and \( x = 2 \). We now evaluate \( g(x) \) at \( x = -2 \), \( x = 0 \), and \( x = 2 \):

\[
\begin{align*}
g(-2) &= (-2)^3 - 3(-2)^2 + 4 = -16 \\
g(0) &= 4 \\
g(2) &= 8 - 12 + 4 = 0
\end{align*}
\]

The minimum value of \( g(x) \) is \(-16\), which occurs at \( x = -2 \); the largest value of \( g(x) \) is 4, which occurs at \( x = 0 \).

Grading: +5 points for finding the critical points, +5 points for evaluating the function at the critical points and endpoints, +5 points for picking out the largest and smallest values. Grading for common mistakes: -2 points if the values of \( x \) were substituted into \( f'(x) \) (or \( g'(x) \)) instead of \( f(x) \) (or \( g(x) \)); -2 points if the values of \( x \) weren’t given, where the largest and smallest function values.
(3) (20 points) A television camera is positioned 5280 ft from the base of a rocket launching pad. The angle of elevation of the camera changes so that the camera remains pointed at the rocket. Assuming that the rocket rises vertically at 650 ft/s, how fast is the camera’s angle of elevation changing when the rocket’s altitude is 3500 ft?

Solution: We start with a diagram (to the left). The variable $x$ represents the height of the rocket, and $\theta$ represents the angle of elevation; we now need a relationship between $x$ and $\theta$. From trigonometry, we know that $\frac{x}{5280} = \tan \theta$. Taking derivatives:

$$\frac{x'}{5280} = \sec^2 \theta \cdot (\theta').$$

Now, $x'$ (which is $\frac{dx}{dt}$) is the rate at which $x$ is changing: this is the speed of the rocket (650 ft/s), and $\theta'$ (or $\frac{d\theta}{dt}$) is the rate at which $\theta$ is changing, which is what we are looking for. We need to find the value of $\sec \theta$. The quickest way to do this is to find the inverse tangent of $\frac{3500}{5280}$, which is approximately 33.53955°.

Then $\sec \theta \approx 1.19976$, so $\frac{650}{5280} = (1.19976)^2 \cdot (\theta')$, and $\theta' \approx 0.0855$.

Grading: +6 points for the trigonometric ratio, +7 points for taking derivatives, +7 points for finding $\theta'$

(3) (20 points) A ladder 12 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 1.5 ft/s, how fast is the angle between the ladder and the wall changing when that angle is $\pi/3$ radians?

Solution: We start with a diagram (to the left). Once again, we need to find a relationship between $x$ and $\theta$. In this case, the relationship for $\sin \theta$ is important:

$$\frac{x}{12} = \sin \theta.$$

Taking derivatives, we have $\frac{x'}{12} = \cos \theta \cdot (\theta')$.

Now we substitute values. The value of $x'$ (which can also be written as $\frac{dx}{dt}$) is how fast the distance from the bottom of the ladder is moving away from the wall, namely 1.5 ft/s; the value of $\theta'$ (or $\frac{d\theta}{dt}$) is how fast the angle $\theta$ is changing, which is the object of the problem. When $\theta = \pi/3$, $\cos \theta = 0.5$. Thus

$$\theta' = \frac{x'}{12 \cos \theta} = \frac{1.5}{(12)(0.5)} = 0.25 \text{ rad/s}.$$

Grading: See N. Grading for common mistakes: −2 points if $\frac{x'}{12}$ was given as the derivative of $\frac{x}{12}$; −13 points for finding $\frac{dy}{dt}$, where $y$ is the distance from the top of the ladder to the ground.
(4) (20 points) Sketch the graph of the function \( f(x) = \left\{ \begin{array}{ll} xe^x - e^x & \\ 2xe^x + e^x & \end{array} \right\} \). Be sure to include any relative maxima, relative minima, and inflection points that may exist. Note that \( \lim_{x \to -\infty} f(x) = 0 \).

Solution: We will find the relative maxima, relative minima, and inflection points first. When \( f(x) = xe^x - e^x \),

\[
\begin{align*}
f'(x) &= xe^x + e^x - e^x = xe^x \\
f''(x) &= xe^x + e^x = (x + 1)e^x.
\end{align*}
\]

Critical points are where \( f'(x) \) is zero or undefined. Since \( f'(x) \) is defined for all \( x \), this means we only have one critical point \( x = 0 \). (Also because \( e^x \) is never zero.) Using a test point, we find that \( f'(x) \) is negative (and \( f(x) \) is decreasing) if \( x < 0 \), and \( f'(x) \) is positive (and \( f(x) \) is increasing) if \( x > 0 \). Hence \( x = 0 \) is a relative minimum.

Now we look for possible inflection points. These points are where \( f''(x) \) is zero or undefined. Once again, \( f''(x) \) is defined for all \( x \), and is zero only if \( x = -1 \). Using test points, we see that \( f(x) \) is concave downward if \( x < -1 \) and concave upward if \( x > -1 \). Hence \( x = -1 \) is an inflection point.

We now plot some important points: \( x \)-intercept(s) \((x = 1)\) is the only \( x \)-intercept, so the point \((1,0)\) is on the graph of \( f(x) \), the critical point \((0,-1)\), and the inflection point \((-1, -e^{-1} - e^{-1}) \approx (-1, -0.735)\). We now “connect the dots”, noticing that \( f(x) \) has a horizontal asymptote of \( y = 0 \) to the left. We get a graph like on the next page, to the left.

For \( S \), the procedure is the same, but the numbers are different.

\[
\begin{align*}
f(x) &= 2xe^x + e^x = (2x + 1)e^x \\
f'(x) &= 2xe^x + e^x \cdot 2 + e^x = (2x + 3)e^x \\
f''(x) &= 2xe^x + 2e^x + 3e^x = (2x + 5)e^x.
\end{align*}
\]

The function \( f(x) \) has an \( x \)-intercept at \( x = -1/2 \), a relative minimum at \( x = -3/2 \), and an inflection point at \( x = -5/2 \). The function \( f(x) \) is also decreasing for all \( x < -3/2 \) and increasing for all \( x > -3/2 \), and is concave downward if \( x < -5/2 \) and concave upward for \( x > -5/2 \). Once again, \( f(x) \) has a horizontal asymptote of \( y = 0 \) to the left, and a sketch of the graph \( f(x) \) appears to the right on the next page. (The coordinates of the special points are \((-3/2, -2e^{-3/2}) \approx (-3/2, -0.446)\) and \((-5/2, -4e^{-5/2}) \approx (-5/2, -0.328)\).)
Grading: +5 points for finding the first and second derivatives of \( f(x) \), +5 points for finding the critical point(s) and the inflection point(s), +10 points for sketching the graph of \( f(x) \). Grading for common mistakes: −2 points for “no inflection points”; −7 points for not finding the second derivative.
(5) (10 points each) Find

(a) the first derivative of \((1 - x)^{(1+x^2)}\).

Solution: The key is to use logarithmic differentiation. First write
\[
y = (1 - x)^{(1+x^2)}.
\]
Then take logarithms:
\[
\ln y = \ln(1 - x)^{(1+x^2)} = (1 + x^2) \ln(1 - x)
\]
Use implicit differentiation:
\[
\frac{1}{y} \cdot y' = (1 + x^2) \cdot \frac{1}{1-x} \cdot (-1) + \ln(1-x) \cdot 2x
\]
\[
y' = y \left[ (1 + x^2) \cdot \frac{1}{1-x} \cdot (-1) + \ln(1-x) \cdot 2x \right]
\]
\[
= (1 - x)^{(1+x^2)} \left[ (1 + x^2) \cdot \frac{1}{1-x} \cdot (-1) + \ln(1-x) \cdot 2x \right]
\]
Grading: +3 points for taking logarithms, +5 points for using implicit differentiation, +2 points for solving for \(y'\) (dy/dx). Grading for common mistakes: +5 points for finding the derivative of \(\ln(y)\).

(b) the second derivative of \(\cosh(3x)\).

Solution: The second derivative of a function is the derivative of the derivative. The easy way to find derivatives of \(\cosh(3x)\) is to convert it to exponential form first:
\[
f(x) = \cosh(3x) = \frac{e^{3x} + e^{-3x}}{2}
\]
\[
f'(x) = \frac{3e^{3x} - 3e^{-3x}}{2}
\]
\[
f''(x) = \frac{9e^{3x} + 9e^{-3x}}{2}
\]
Answer: \(\frac{9e^{3x} + 9e^{-3x}}{2}\), which is the same as \(9 \cosh(3x)\).
Grading: +5 points for each derivative.

(c) \(\lim_{x \to e} \frac{\ln x - 1}{x - e}\).

Solution: Trying to put in \(x = e\) results in a limit of the type \(0/0\), so the limit is an indeterminate form. Trying L’Hospital’s rule, we see that
\[
\lim_{x \to e} \frac{\ln x - 1}{x - e} \overset{\text{L'H}}{=} \lim_{x \to e} \frac{1/x}{1} = \frac{1}{e}.
\]
Grading: +3 points for recognizing the indeterminate form, +5 points for trying L’Hospital’s rule, +2 points for evaluating the second limit.
(a) the first derivative of \((1 - x^2)^{(1+x)}\).

**Solution:** The key is to use logarithmic differentiation. First write

\[ y = (1 - x^2)^{(1+x)}. \]

Then take logarithms:

\[ \ln y = \ln(1 - x^2)^{(1+x)} = (1 + x) \ln(1 - x^2) \]

Use implicit differentiation:

\[ \frac{1}{y} \cdot y' = (1 + x) \cdot \frac{1}{1 - x^2} \cdot (-2x) + \ln(1 - x^2) \]

\[ y' = y \left[ (1 + x) \cdot \frac{1}{1 - x^2} \cdot (-2x) + \ln(1 - x^2) \right] \]

\[ = (1 - x^2)^{(1+x)} \left[ (1 + x) \cdot \frac{1}{1 - x^2} \cdot (-2x) + \ln(1 - x^2) \right] \]

Grading: +3 points for taking logarithms, +5 points for using implicit differentiation, +2 points for solving for \(y'\) (\(dy/dx\)). Grading for common mistakes: +5 points for finding the derivative of \(\ln(y)\).

(b) the second derivative of \(\sinh(2x)\).

**Solution:** The second derivative of a function is the derivative of the derivative. The easy way to find derivatives of \(\sinh(2x)\) is to convert it to exponential form first:

\[ f(x) = \sinh(2x) = \frac{e^{2x} - e^{-2x}}{2} \]

\[ f'(x) = \frac{2e^{2x} + 2e^{-2x}}{2} \]

\[ f''(x) = \frac{4e^{2x} - 4e^{-2x}}{2} \]

Answer: \(\frac{4e^{2x} - 4e^{-2x}}{2}\), which is the same as \(4 \sinh(3x)\).

Grading: +5 points for each derivative.

(c) \(\lim_{x \to 0} \frac{e^x - 1}{x^2 + x}\).

**Solution:** Trying to put in \(x = 0\) results in a limit of the type 0/0, so the limit is an indeterminate form. Trying L’Hospital’s rule, we see that

\[ \lim_{x \to 0} \frac{e^x - 1}{x^2 + x} \leq \lim_{x \to 0} \frac{\frac{e^x}{2x + 1}}{2x + 1} = \frac{1}{1} = 1. \]

Grading: +3 points for recognizing the indeterminate form, +5 points for trying L’Hospital’s rule, +2 points for evaluating the second limit.