Solutions to MAT 270 Test #1

Because there are two versions of the test, solutions will only be given for Form N (\([N]\)). Differences from the Form S (\([S]\)) version will be given. (The values for Form N appear above those of Form S in the curly braces \{\}).

\(N\) \(S\) (1) (20 points) **Using the definition of the derivative**, find \(f'(a)\) where \(f(x) = \begin{cases} x^2 - 1 \\ x^2 + 4 \end{cases}\) and \(a = \begin{cases} 3 \\ 1 \end{cases}\).

**Solution:**

One definition for the derivative (used here with \([N]\)) is

\[
f'(3) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{(3 + h)^2 - 1 - (3^2 - 1)}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} (6 + h) = 6 + 0 = 6.
\]

The other version of the derivative could also have been used. It is used here with the answer to \([S]\):

\[
f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(x^2 + 4) - (1^2 + 4)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2.
\]

Grading for common mistakes: -10 points if neither definition was used; -1 point if \(f'(x)\) was given as a formula, but \(f'(a)\) wasn’t found explicitly; -5 points if the lim was missing.

\(N\) (2) (15 points) Evaluate \(\lim_{x \to +\infty} \frac{2x^2 + x - 1}{3x^2 - 5}\).

**Solution:** The function is a rational function, and \(x\) is approaching \(\pm\infty\), so we need to factor out the highest power of \(x\) in the numerator and in the denominator and look for cancellation:

\[
\lim_{x \to +\infty} \frac{2x^2 + x - 1}{3x^2 - 5} = \lim_{x \to +\infty} \frac{x^2 (2 + \frac{1}{x} - \frac{1}{x^2})}{x^2 (3 - \frac{5}{x^2})} = \lim_{x \to +\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{3 - \frac{5}{x^2}} = \frac{2}{3}.
\]

Grading for common mistakes: -5 points if there was no work; -3 points for an answer of \(\infty\) \([S]\).
(2) (15 points) Evaluate \( \lim_{x \to -\infty} \frac{2x^3 + x - 3}{3x^2 + 8x - 1} \).

Solution: We proceed as in [N].

\[
\lim_{x \to -\infty} \frac{2x^3 + x - 3}{3x^2 + 8x - 1} = \lim_{x \to -\infty} \frac{x^3 \left( 2 + \frac{1}{x^2} - \frac{3}{x^3} \right)}{3 + \frac{8}{x} - \frac{1}{x^2}} = \lim_{x \to -\infty} \frac{x \left( 2 + \frac{1}{x^2} - \frac{3}{x^3} \right)}{3 + \frac{8}{x} - \frac{1}{x^2}}
\]

The denominator is approaching 3; and the numerator is the product of a number approaching \(-\infty\) and a number near 2, so the numerator is approaching \(-\infty\) as well. Since a number approaching \(-\infty\) divided by a number approaching 3 is a number approaching \(-\infty\), the limit is \(-\infty\).

Grading: See [N].

(3) (15 points) Use the precise definition of a limit to show that \( \lim_{y \to 3} (2y - 1) = 5 \).

Solution: The precise definition of a limit is: \( \lim_{y \to a} f(y) = L \) if, for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that if \( 0 < |y - a| < \delta \), then \( |f(y) - L| < \varepsilon \).

You need to find the relationship between \( |y - a| \) and \( |f(y) - L| \). \( |y - a| = |y - 3| \), and

\[
|f(y) - L| = |(2y - 1) - 5| = |2y - 6| = |2(y - 3)| = 2 \cdot |y - 3|.
\]

Hence, to have \( |f(y) - L| < \varepsilon \), we must have

\[
\delta = |y - a| = |y - 3| = \frac{2 \cdot |y - 3|}{2} < \frac{\varepsilon}{2},
\]

so we can choose \( \delta = \frac{\varepsilon}{2} \).

Grading: +5 points for work, +10 points for finding \( \delta \). Grading for common mistakes: -8 points if the precise definition was not used.

(3) (15 points) Use the precise definition of a limit to show that \( \lim_{y \to 2} (3y + 1) = 7 \).

Solution: The explanation is the same as in version [N]: here, \( |y - a| = |y - 2| \), and

\[
|f(y) - L| = |(3y + 1) - 7| = |3y - 6| = |3(y - 2)| = 3 \cdot |y - 2|,
\]

so we can guarantee that \( |f(y) - L| < \varepsilon \) if

\[
\delta = |y - a| = |y - 2| = \frac{|f(y) - L|}{3} < \frac{\varepsilon}{3}.
\]

Grading: See [N].
(4) (20 points) A function $g(x)$ is given below. Where is $g(x)$ continuous? Justify your answer.

$$g(x) = \begin{cases} 
  e^{x^{-1}}, & \text{if } x > 1; \\
  x, & \text{if } 0 < x \leq 1; \\
  \sqrt{1-x}, & \text{if } x \leq 0.
\end{cases}$$

**Solution:** This is a piecewise-defined function, with $g(x)$ defined to be one function over an interval, another function over another, etc. The functions $e^{x^{-1}}$ and $x$ are continuous everywhere, so $g(x)$ is continuous at the points in the interior of the intervals where $g(x) = e^{x^{-1}}$ and $g(x) = x$ ($(1, +\infty)$ and $(0, 1)$, respectively). $\sqrt{1-x}$ is continuous where $1-x > 0$, i.e., where $x < 1$, which is everywhere in the interior of the interval where $g(x) = \sqrt{1-x}$ $((-\infty, 0))$.

The only places left to check for continuity are $x = 0$ and $x = 1$. To do this algebraically, we need to see whether $\lim_{x \to a} g(x) = g(a)$ for $a = 0$ and $a = 1$. We calculate some limits:

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x = 0$$
$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \sqrt{1-x} = \sqrt{1-0} = 1$$
$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} e^{x^{-1}} = 1$$
$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x = 1$$

Since the first limit does not exist, $g(x)$ is not continuous at 0. Since the second limit exists and is equal to $g(1)$, $g(x)$ is continuous at 1. Hence, $g(x)$ is continuous at all real numbers except 0.

Grading: Full credit was awarded if the graph of $g(x)$ was sketched, and continuity based on that sketch.

Grading: +5 points for each limit, +5 points for the “nearsighted ant” analysis, +5 points for the final answer.

(4) (20 points) A function $g(x)$ is given below. Where is $g(x)$ continuous? Justify your answer.

$$g(x) = \begin{cases} 
  \ln x, & \text{if } x > 0; \\
  -x, & \text{if } -1 < x \leq 0; \\
  e^{x+1}, & \text{if } x \leq -1.
\end{cases}$$

**Solution:** The analysis is explained in depth in version N. The function $g(x)$ is continuous at every $x$ in the intervals $(-1, 0)$ (where $g(x) = -x$, which is continuous everywhere), $(-\infty, -1)$ (where $g(x) = e^{x+1}$, which is continuous everywhere),
and $\mathbb{R} \cup (0, +\infty)$ (where $g(x) = \ln x$, which is continuous on the same interval). We need to see what $g(x)$ does as $x$ approaches $-1$ and $0$:

\[
\begin{align*}
\lim_{x \to -1^-} g(x) &= \lim_{x \to -1^-} e^{x+1} = e^0 = 1 \\
\lim_{x \to -1^+} g(x) &= \lim_{x \to -1^+} (-x) = 1
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to 0^-} g(x) &= \lim_{x \to 0^-} (-x) = 0 \\
\lim_{x \to 0^+} g(x) &= \lim_{x \to 0^+} \ln x = -\infty
\end{align*}
\]

So $g(x)$ is continuous at $-1$ but not at $0$, and hence $g(x)$ is continuous for all real numbers except $0$.

Grading: See N.

\[\boxed{N} (5) \text{ (15 points) Find } \lim_{t \to 1} (e^{-t} + \ln t).\]

Solution: The function $e^{-t} + \ln t$ is continuous at $1$, so this limit is just $e^{-1} + \ln 1 = \frac{1}{e} \approx 0.367879$.

Grading for common mistakes: -8 points for finding the derivative; +5 points (total) for miscellaneous work.

\[\boxed{S} (5) \text{ (15 points) Find } \lim_{t \to 0} e^{t+1} \cos(2t).\]

Solution: The function $e^{t+1} \cos(2t)$ is continuous at $t = 0$, so the limit is just $e^{0+1} \cos(2 \cdot 0) = e \approx 2.71828$.

Grading: See N.
The following is the graph of a function $F(x)$. Sketch the graph of $F'(x)$.

Solution: Sketches are provided. The horizontal lines in the figures below are really horizontal lines; lines which are not horizontal could actually be curved.

Grading: This problem was graded on a 0–5–10–15 basis: +15 points were given if the essential features of $F'(x)$ were as shown, except possibly at one point; +10 points were given if it was clear that principles about sketching $F'(x)$ (increasing/decreasing, relative minimums/maximums, places where $F'(x)$ does not exist, etc) were used; +5 points for miscellaneous graphs.