The exam has five problems and is five pages long. **You must do problem #1 and three of the remaining four problems.** Be sure to indicate which problem you do not want counted. Answers without sufficient work shown will not receive any points. If you are having difficulty with a problem, you should skip it and work on another. Note that all numbers are decimal (base 10).

(1) Find a recurrence relation for the number of bit strings of length $n$ that contain a pair of consecutive 0s. You do not need to solve this recurrence relation. (Hint: Consider the following cases: the last bit is a 1, the last two bits are 10, or the last two bits are 00. Don’t forget the initial conditions.) (25 points)
(2) Solve the following recurrence relation. (25 points)

\[ d_n - 6d_{n-1} - 16d_{n-2} = 2^n, \quad n \geq 2, \]
\[ d_0 = 1, \]
\[ d_1 = 5. \]
Recall that an *onto* function is a function $f : A \to B$ such that for every element $b \in B$, there exists an element $a \in A$ with $f(a) = b$. In this problem we will count the number of onto functions from $A$ to $B$, where $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3\}$.

(a) How many functions $g : S \to T$ are there, if $S$ has $m$ elements and $T$ has $n$ elements? (5 points)

Let the set $A_i$ consist of all functions $f : A \to B$ such that $i$ is not in the range of $f$, where $i$ is 1, 2, or 3. (For instance, $A_1$ is the set of all functions $f$ from $A$ to $B$ which do **not** have 1 in the range of $f$; i.e., $f$ is then a function from $A$ to $B \setminus \{1\} = \{2, 3\}$.)

(b) What is $|A_1 \cup A_2 \cup A_3|$? (15 points)

(c) What does the set $A_1 \cup A_2 \cup A_3$ have to do with onto functions from $A$ to $B$? How many onto functions are there from $A$ to $B$? (5 points)
(4) How many integers between 0 and 9,999,999 (inclusive) are there whose digits add up to 23? (25 points)
(5) Suppose that $C_n$ satisfies the homogeneous linear recurrence relation

$$C_{n+5} = 5C_{n+4} - 7C_{n+3} - C_{n+2} + 8C_{n+1} - 4C_n, \quad \text{for } n \geq 5.$$ 

What is the general form for the solution to this recurrence? (Hint: The auxiliary equation for this recurrence is $(r + 1)(r - 1)^2(r - 2)^2 = 0.$) (25 points)