Solutions to MAT 117 Test #3

Because there are two versions of the test, solutions will only be given for Form A. Differences from the Form B version will be given. (The values for Form A appear above those of Form B in the curly braces { }.)

Part I: Answer the Questions and Show All Your Work

1. [10 pts] Algebraically solve the following exponential equation. (That is, find exact solutions.)

   Form A
   \[ e^{2x} + 3e^x - 18 = 0 \]

   Solution: Since \( e^x \) and \( e^{2x} \) both appear in this equation, the substitution \( y = e^x \) should be made. Doing this produces the equation \( y^2 + 3y - 18 = 0 \), which factors into \((y + 6)(y - 3) = 0\), so \( y = -6 \) or \( y = 3 \). We need to find \( x \) for these values of \( y \). Since \( y = e^x \), \( x = \ln y \), so we have the solutions \( x = \ln(-6) \) and \( x = \ln 3 \). The first of these values is not defined, so in fact there is only one solution: \( x = \ln 3 \).

   Grading: +3 points for the substitution \( y = e^x \), +4 points for finding \( y \), +3 points for finding \( x \). Grading for common mistakes: -2 points for including \( \ln(-6) \) (or \( \ln(-3) \)) as a solution; +5 points (total) for some work involving logarithms; -3 points for \( x = 6 \) (or \( x = 5 \) and 3); -3 points for an approximation.

   Form B
   \[ e^{2x} - 2e^x - 15 = 0 \]

   Solution: The procedure is the same as for Form A. If \( y = e^x \), then \( y^2 - 2y - 15 = 0 \), and this equation factors into \((y - 5)(y + 3) = 0\). Hence \( y = 5 \) or \( y = -3 \), and \( x = \ln 5 \) or \( x = \ln(-3) \). Once again, the second “solution” is not defined, so there is only one solution: \( x = \ln 5 \).

   Grading: See Form A.
2. [10 pts] Algebraically solve the following logarithm equation.

Form A \[ \log_{10} x + \log_{10}(x - 3) = 1 \]

**Solution:** Since there are two logarithm functions in this equation, we should try to combine them.* We can do this, and we get the equation \[ \log_{10} x(x - 3) = 1 \]. The exponential form of this equation is \[ 10^1 = x(x - 3) \], which is a quadratic equation. Multiplying out this equation and moving all the terms to the right-hand side, we get \[ x^2 - 3x - 10 = 0 \]. This equation factors into \( (x - 5)(x - 2) = 0 \), so \( x = 5 \) or \( x = 2 \). We need to test both of these “alleged” solutions, and only \( x = 5 \) turns out to be a solution (since \( \log_{10}(-2) \) is not defined).

Grading: +3 points for combining logarithms, +4 points for solving the quadratic equation, +3 points for testing and giving a final answer. Grading for common mistakes: −2 points for including \( x = -2 \).

Form B \[ \log_6 x + \log_6(x + 1) = 1 \]

**Solution:** We proceed the same as in Form A. Combining logarithms, we get \( \log_6 x(x + 1) = 1 \), so \( 6^1 = x(x + 1) \). This is a quadratic equation, which is \( x^2 + x - 6 = 0 \) after moving all the terms to the right-hand side. It also factors into \( (x - 2)(x + 3) = 0 \), so \( x = 2 \) or \( x = -3 \), but \( x = -3 \) is not a solution, because \( \log_6(-2) \) is not defined. (Any logarithm of a negative number is undefined.) So the only solution is \( x = 2 \). (This can be verified using your calculator, if you use the change-of-basis formula: \( \log_6 2 = \frac{\log_{10} 6}{\log_{10} 2} \).)

Grading: See Form A.

3. [10 pts] Find a third degree polynomial with real coefficients that has \{ 2, −3 \} and \{ i, 4i \} as roots (zeros).

**Solution:** A polynomial that has 2 and \( i \) as roots is \( (x - 2)(x - i) \), but this polynomial does not have real coefficients. To get a polynomial \( p(x) \) with real coefficients, \( p(x) \) must also have the conjugate of \( i \) (i.e., \( -i \)) as a root. Hence, we can let \( p(x) = (x - 2)(x - i)(x + i) = (x - 2)(x^2 + 1) \); any multiple of \( (x - 2)(x^2 + 1) \) would also be correct.

For Form B, any multiple of \( (x - 3)(x - 4i)(x + 4i) \) is a correct answer.

Grading: Partial credit: +3 points for any work involving polynomials; +7 points (total) if \( -i \) (or \( -4i \), for Form B) was not included as a root.

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* Note that \( \log_{10}(x - 3) \) is NOT the same as \( \log_{10} x - \log_{10} 3 \), nor is it anything nice.
4. [7 pts] The growth in the percentage of college freshmen who reported involvement in volunteer work during their last year of high school is approximated by

\[ f(t) = \begin{cases} 
-662.1 + 154 \ln(t + 95) \\
-663.3 + 161 \ln(t + 80) 
\end{cases}, \quad 0 \leq t \leq 25, \]

where \( t \) represents the number of years since 1975, and \( f(t) \) is the percentage. According to the above function, during what year was the percentage \( \left\{ \frac{70}{80} \right\} \)%?

Solution: For Form A, the value of \( t \) such that \( f(t) = 70 \) needs to be found. This can be done by trial and error (or graphing the function \( f(t) \) and using the TRACE or ZOOM functions), or the equation can be solved algebraically:

\[-662.1 + 154 \ln(t + 95) = 70 \]
\[154 \ln(t + 95) = 70 + 662.1 = 832.1 \]
\[\ln(t + 95) = \frac{832.1}{154} \]
\[t + 95 = e^{\frac{832.1}{154}} \]
\[t = e^{\frac{832.1}{154}} - 95 \approx 21.035.\]

To find the year when \( f(t) = 70 \), we round 21.035 down, then add that number to 1975: The year was 1975 + 21 = 1996.

For Form B, the procedure is the same: We want to find the value of \( t \) so that \( f(t) = 80 \):

\[-663.3 + 161 \ln(t + 80) = 80 \]
\[161 \ln(t + 80) = 80 + 663.3 = 743.3 \]
\[\ln(t + 80) = \frac{743.3}{161} \]
\[t + 80 = e^{\frac{743.3}{161}} \]
\[t = e^{\frac{743.3}{161}} - 80 \approx 21.167.\]

Hence the answer to Form B is also 1975 + 21 = 1996.

Grading: +3 points for writing the equation \( f(t) = 70 \) (or 80), +3 points for solving for \( t \), +1 point for the year. Grading for partial credit: +3 points (total) for \( f(70) \) or \( f(0.70) \) (Form A), \( f(80) \) or \( f(0.80) \) (Form B).
5. The population of a city is given by the equation \( P = \begin{cases} 150,200e^{0.011t} \\ 130,250e^{0.010t} \end{cases} \), where \( t \) is the time in years, with \( t = 0 \) corresponding to 1990.

a. [5 pts] Find the population of the city in the year \( \begin{cases} 2008 \\ 2007 \end{cases} \)

Solution: Form A: The year 2008 corresponds to \( t = 2008 - 1990 = 18 \), so the population in 2008 is \( P \) when \( t = 18 \), or 183,088. For Form B, \( t = 2007 - 1990 = 17 \), and \( P = 154,386 \).

Grading: +2 points for finding \( t \), +2 points for substituting \( t \), +1 point for the final answer.

b. [5 pts] During what year will the city have a population of \( \begin{cases} 200,000 \\ 250,000 \end{cases} \)?

Solution: For Form A, we need to find the value of \( t \) so that \( P = 200,000 \); in other words, we need to solve the equation

\[
150,200e^{0.011t} = 200,000 \\
e^{0.011t} = \frac{200,000}{150,200} \\
0.011t = \log_e \left( \frac{200,000}{150,200} \right) \\
t = \frac{1}{0.011} \ln \left( \frac{200,000}{150,200} \right) \approx 26.03,
\]

so the population will reach 200,000 in the year \( 1990 + 26 = 2016 \).

Form B: We need to solve the equation

\[
130,250e^{0.010t} = 250,000 \\
e^{0.010t} = \frac{250,000}{130,250} \\
0.010t = \log_e \left( \frac{250,000}{130,250} \right) \\
t = \frac{1}{0.010} \ln \left( \frac{250,000}{130,250} \right) \approx 65.2,
\]

so the population will reach 250,000 in the year \( 1990 + 65 = 2055 \).

Grading: +2 points for the equation \( P = 200,000 \) (or 250,000), +2 points for solving for \( t \), +1 point for finding the year. Grading for common mistakes: −2 points for assuming the growth is linear.
6. \[\text{[6 pts] Simplify } \begin{cases} \ln \frac{e^{x+5}}{e^2} \\ \ln \frac{e}{e^{y+1}} \end{cases}.\]

\textbf{Solution: Form A:}

\[
\ln \frac{e^{x+5}}{e^2} = \ln e^{x+5} - \ln e^2 = (x + 5) \ln e - 2 \ln e = (x + 5) - 2 = x + 3, \text{ or }
\]

\[
\ln \frac{e^{x+5}}{e^2} = \ln e^{(x+5)-2} = \ln e^{x+3} = x + 3;
\]

\textbf{Form B:}

\[
\ln \frac{e}{e^{y+1}} = \ln e - \ln e^{y+1} = \ln e - (y + 1) \ln e = 1 - (y + 1) = -y, \text{ or }
\]

\[
\ln \frac{e}{e^{y+1}} = \ln e^{1-(y+1)} = \ln e^{-y} = -y.
\]

\text{Grading: This problem required that two properties of logarithms be used; +3 points were given for each of these two properties. Grading for common mistakes: +5 points (total) for } \frac{x + 5}{2} \text{ (or } \frac{1}{y + 1}); +3 \text{ points (total) for } y + 1 \text{ (Form B).}

7. \[\text{[6 pts] Write the following as the logarithm of a single quantity: } \begin{cases} 2 \log_5 x - 3 \log_5 y \\ 3 \log_3 y - \log_3 (z + 1) \end{cases}.\]

\textbf{Solution: Form A:}

\[
2 \log_5 x - 3 \log_5 y = \log_5 x^2 - \log_5 y^3 = \log_5 \left( \frac{x^2}{y^3} \right);
\]

\textbf{Form B:}

\[
3 \log_3 y - \log_3 (z + 1) = \log_3 y^3 - \log_3 (z + 1) = \log_3 \left( \frac{y^3}{z + 1} \right).
\]

\text{Grading: This problem required that two properties of logarithms be used; +3 points were given for each of these two properties. Grading for common mistakes: +4 points for } \frac{2 \log_5 x}{3 \log_5 y} \text{ (Form A) or } \frac{3 \log_3 y}{\log_3 (z + 1)} \text{ (Form B).}
8. [6 pts] Approximate \( \left\{ \frac{\log_3 8}{\log_4 9} \right\} \). Give at least three decimal digits.

**Solution:** The change of basis formula needs to be used, since your calculator doesn’t do logarithms in base 3 or 4. Form A:

\[
\log_3 8 = \frac{\log_{10} 8}{\log_{10} 3} \approx 1.893, \text{ or } \\
\log_3 8 = \frac{\ln 8}{\ln 3} \approx 1.893. \\
\]

Form B:

\[
\log_4 9 = \frac{\log_{10} 4}{\log_{10} 9} \approx 1.585, \text{ or } \\
\log_4 9 = \frac{\ln 4}{\ln 9} \approx 1.585. \\
\]

**Grading:** +3 points for the change of basis formula, +3 points for finding the approximation. 

**Grading for common mistakes:** +4 points (total) for \( \frac{\log_{10} 8}{\log_{10} 3} \) or \( \frac{\ln 8}{\ln 3} \) (Form A), or \( \frac{\log_{10} 4}{\log_{10} 9} \) or \( \frac{\ln 4}{\ln 9} \) (Form B).
Part II: Circle the Correct Choice

Each problem is worth 5 points.

9. Find the equation(s) of the vertical asymptote(s) of \( f(x) = \begin{cases} \frac{x^2 - 3}{x - 3} \\ \frac{x - 3}{3x + 2} \\ \frac{3x + 2}{x^2 + 16} \end{cases} \).

Answer: The vertical asymptotes of \( f(x) \) are the values of \( x \) that make the denominator zero. Hence, for Form A, \( x - 3 = 0 \), or \( x = 3 \) (choice (b)); for Form B, \( x^2 + 16 = 0 \), which has no real solutions (choice (d)).

10. Find the equation(s) of the horizontal asymptote(s) (if any) of \( f(x) = \begin{cases} \frac{x^2 - 3}{x - 3} \\ \frac{x - 3}{3x + 2} \\ \frac{3x + 2}{x^2 + 16} \end{cases} \).

Answer: If \( x \) is large in absolute value, \( f(x) \approx \frac{x^2}{x} = x \), so \( f(x) \) has no horizontal asymptotes for Form A (choice (d)). For Form B, \( f(x) \approx \frac{3x}{x^2} = \frac{1}{x} \). Since \( \frac{1}{x} \) gets close to 0 as \( x \) gets big in absolute value, \( f(x) \) has \( y = 0 \) as a horizontal asymptote, choice (a).

11. Find the amount in an account, where \( \begin{cases} \$10,000 \\ \$15,000 \end{cases} \) was invested at a rate of \( \begin{cases} 7\% \\ 8\% \end{cases} \) for \( \begin{cases} 8 \text{ years} \\ 10 \text{ years} \end{cases} \), if the interest is compounded quarterly.

Answer: Form A: \( 10,000 \left(1 + \frac{0.07}{4}\right)^{4 \cdot 8} \approx 17,422.13 \), choice (a). Form B: \( 15,000 \left(1 + \frac{0.08}{4}\right)^{4 \cdot 10} \approx 33,120.59 \), choice (c).

12. Find the balance (in problem 11) if the money is compounded continually instead.

Answer: Form A: \( 10,000e^{0.07 \cdot 8} \approx 17,506.73 \), choice (b); Form B: \( 15,000e^{0.08 \cdot 10} \approx 33,383.11 \), choice (c).
13. Find the domain of the function \( f(x) = \left\{ \begin{array}{ll} -3 + \log_2(2x - 3) & \\
-2 \log_2(4 - 2x) & \end{array} \right\} \).

Answer: The rule for domains that applies here is that you cannot take the logarithm of a negative number, or of zero. So, in Form A, we must have \( 2x - 3 > 0 \), or \( x > \frac{2}{3} \), answer (d); in Form B, we must have \( 4 - 2x > 0 \), or \( x < 2 \), answer (c).

14. Find the \( x \)-intercept of the function \( f(x) = \left\{ \begin{array}{ll} -3 + \log_2(2x - 3) & \\
-2 \log_2(4 - 2x) & \end{array} \right\} \).

Answer: The \( x \)-intercept(s) of a function \( f(x) \) are the values of \( x \) such that \( f(x) = 0 \). Hence, for Form A, we must have \(-3 + \log_2(2x - 3) = 0\). Solving for \( x \), or trying the choices available (and using logarithm properties, or the change of basis formula) we see that \( x = \frac{11}{2} \), choice (c).

For Form B, we must have \(-2 \log_2(4 - 2x) = 0\), and this is only true if \( x = \frac{3}{2} \), choice (a).

15. Evaluate \( \left\{ \frac{\log_a 24}{\log_a 36} \right\} \), if \( \log_a 2 = 0.6055 \) and \( \log_a 3 = 0.9597 \).

Answer: Form A:

\[ \log_a 24 = \log_a (2^3 \cdot 3) = 3 \log_a 2 + \log_a 3 = 3(0.6055) + (0.9597) = 2.7762, \]

answer (b). For Form B,

\[ \log_a 36 = \log_a (2^2 \cdot 3^2) = 2 \log_a 2 + 2 \log_a 3 = 2(0.6055) + 2(0.9597) = 3.1304, \]

answer (c).

MAT 117 Website: \url{http://math.asu.edu/~checkman/117/}