Solutions to MAT 117 Test #2

Because there are two versions of the test, solutions will only be given for Form A. Differences from the Form B version will be given. (The values for Form A appear above those of Form B in the curly braces { }.)

Part I: Answer the Questions and Show All Your Work

1. Your hourly wage \( y \) in terms of the number of units you produce \( x \) is given by \( y = f(x) = \begin{cases} 8 + 0.35x \\ 7 + 0.3x \end{cases} \).

   a. [5 pts] Algebraically find \( f^{-1}(x) \) (the inverse of the function \( f(x) \)).

   **Solution:** To find \( f^{-1}(x) \), take the equation which relates \( x \) and \( y \) (\( y = 8 + 0.35x \) for Form A), exchange the \( x \) and \( y \), and solve the resulting equation for \( y \):
   
   \[
   \begin{align*}
   y &= 8 + 0.35x \\
   x &= 8 + 0.35y \\
   x - 8 &= 0.35y \\
   \frac{x - 8}{0.35} &= y,
   \end{align*}
   \]
   
   So \( f^{-1}(x) = \frac{x - 8}{0.35} \).

   For Form B, the procedure is the same, except the steps are:
   
   \[
   \begin{align*}
   y &= 7 + 0.3x \\
   x &= 7 + 0.3y \\
   x - 7 &= 0.3y \\
   \frac{x - 7}{0.3} &= y,
   \end{align*}
   \]
   
   And the inverse of \( f(x) \) is \( f^{-1}(x) = \frac{x - 7}{0.3} \).

   **Grading:** +3 points for exchanging \( x \) and \( y \) in the formula, +2 points for solving for \( y \). Grading for common mistakes: +2 points (total) for \( -f(x) \) instead of \( f^{-1}(x) \).

   b. [5 pts] Find and interpret \( f^{-1}\left(\begin{array}{c} 10 \\ 9 \end{array}\right) \) in the context of the problem.

   **Solution:** To find \( f^{-1}(10) \), take the function you found in (a) and substitute \( x = 10 \):
   
   \[
   f^{-1}(10) = \frac{10 - 8}{0.35} = \frac{40}{7} \approx 5.71, \quad \text{for Form A};
   \]

   \[
   f^{-1}(9) = \frac{9 - 7}{0.3} = \frac{20}{3} \approx 6.67, \quad \text{for Form B}.
   \]

   In either case, \( f^{-1}(y) \) is the number of units you need to produce to earn $\( y \) (per hour).

   **Grading:** +2 points for substituting 10 (or 9) for \( x \) in the function from (a), +3 points for the interpretation.
2. Given that \( g(x) = \begin{cases} 3x^2 - 6x - 10 \\ 2x^2 + 8x + 7 \end{cases} \),

a. [5 pts] Write \( g(x) \) in the form \( g(x) = a(x - h)^2 + k \).

Solution: The key to the problem is to complete the square:

\[
g(x) = 3x^2 - 6x - 10 = 3(x^2 - 2x) - 10 = 3(x^2 - 2x + 1) - 10 - 3 \cdot 1 = 3(x - 1)^2 - 13, \text{ for Form A;}
\]

\[
g(x) = 2x^2 + 8x + 7 = 2(x^2 + 4x) + 7 = 2(x^2 + 4x + 4) + 7 - 2 \cdot 4 = 2(x + 2)^2 - 1, \text{ for Form B.}
\]

Grading: +3 points for adding the 1 or 4 inside the parentheses, +2 points for factoring the result. Grading for common mistakes: −1 points for the right answer without work; −3 points for the wrong answer with no work; −1 points for having \( a \) in front \((a(x - 1)^2 - 13, \text{ for instance})\).

b. [5 pts] Find the roots of \( g(x) \).

Solution: The function \( g(x) \) is a quadratic function, so the quadratic equation can be used, or the roots can be approximated (by zooming or tracing the graph of \( g(x) \)), since it was not stated that you have to do this algebraically.* For Form A,

\[
x = \frac{6 \pm \sqrt{(-6)^2 - 3 \cdot 4 \cdot (-10)}}{6} = \frac{6 \pm \sqrt{156}}{6} \approx 3.082 \text{ or } -1.082.
\]

For Form B,

\[
x = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 2 \cdot 7}}{4} = \frac{-8 \pm \sqrt{8}}{4} \approx -1.293 \text{ or } -2.707.
\]

Grading for partial credit: +3 points for using the Rational Root Theorem.

* Finding solutions algebraically means finding exact (not approximate) answers, via manipulation of equations.
3. [10 pts] Algebraically, find all solutions to the given equation:
\[
\begin{align*}
\{ & x^4 - x^3 - 7x^2 + 5x + 10 \\
& x^4 - 2x^3 - 6x^2 + 14x - 7 \}\ = 0
\end{align*}
\]

Solution: Since the solutions to the equation are the roots of a polynomial \( p(x) \), you need to go through the process of finding roots, via the “Good” (zero), the “Bad” (rational numbers), and the “Ugly” (irrational numbers). Zero is not a root of either polynomial, so the next step is to find the possible rational roots.

For Form A, this means looking at all fractions \( \frac{p}{q} \), where \( p \) divides evenly into \( 10 \) and \( q \) divides evenly into \( 1 \). The candidates are \( \pm 10, \pm 5, \pm 2, \) and \( \pm 1 \). Testing the roots shows that \(-1\) and \(2\) are roots. To find the remaining roots, we divide \( p(x) \) by \( x - 2 \) and divide that quotient by \( x + 1 \). Using synthetic division, we have:

\[
\begin{array}{c|cccc}
-1 & 1 & -1 & -7 & 5 & 10 \\
& -1 & 2 & 5 & -10 \\
\hline
1 & -2 & -5 & 10 \\
\end{array}
\quad \text{and then} \quad
\begin{array}{c|cccc}
2 & 1 & -2 & -5 & 10 \\
& 2 & 0 & -10 \\
\hline
1 & 0 & -5 \\
\end{array}
\]

The quotient of the second division is \( x^2 - 5 \), and the “Ugly” roots (\( \pm \sqrt{5} \)) can be found with the quadratic formula. The roots of \( p(x) \) are then \(-1\), \(2\), and \( \pm \sqrt{5} \).

For Form B, the process is the same, except the Rational Root Test yields the candidates \( \pm 1 \) and \( \pm 7 \). Only \(1\) turns out to be a root, but it has a multiplicity of \(2\), because the quotient obtained from dividing \( p(x) \) by \( x - 1 \) also has \(1\) as a root. The synthetic divisions are:

\[
\begin{array}{c|cccc}
1 & 1 & -2 & -6 & 14 & -7 \\
& 1 & -1 & -7 & 7 \\
\hline
1 & -1 & -7 & 7 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|cccc}
1 & 1 & -1 & -7 & 7 \\
& 1 & 0 & -7 \\
\hline
1 & 0 & -7 \\
\end{array}
\]

and the quotient from the second division is \( x^2 - 7 \), which has the roots \( \pm \sqrt{7} \). The roots of \( p(x) \) are thus \(1\) (multiplicity \(2\)) and \( \pm \sqrt{7} \).

Grading: +3 points for the Rational Root Test, +4 points for the divisions, +3 points for finding the solutions to \( x^2 - 5 = 0 \) or \( x^2 - 7 = 0 \). Grading for common mistakes: +5 points for one root; +7 points for two roots.
4. An open box is to be made from a rectangular piece of material which is \( \begin{bmatrix} 16 \\ 14 \end{bmatrix} \) inches by \( \begin{bmatrix} 8 \\ 10 \end{bmatrix} \) inches, by cutting equal squares (of size \( x \)) from each of the corners and then folding up the sides.

a. [5 pts] Write the volume \( V \) of the box as a function of \( x \).

Solution: Drawing a diagram helps:

For Form A, when the corners are cut out and the sides folded up, the height of the box will be \( x \), the width (left-right) will be \( 8 - 2x \), and the length (up-down) will be \( 16 - 2x \). The volume of a box is length times width times height, so the volume will be \( V = (8 - 2x)(16 - 2x)x \).

For Form B, the width will be \( 10 - 2x \) and the length \( 14 - 2x \), and the volume will be \( V = (10 - 2x)(14 - 2x)x \).

Grading for partial credit: +3 points total for any volume formula.

b. [5 pts] Find a value of \( x \) that makes the volume equal to \( \begin{bmatrix} 60 \\ 48 \end{bmatrix} \) cubic inches.

Solution: For Form A, this involves solving the equation \((8 - 2x)(16 - 2x)x = 60\), or

\[(4 - x)(8 - x)x = 15,\]

which is easier to solve, because the Rational Root Test doesn’t provide as many candidates for roots; after some work, it is found that \( x = 3 \) makes the volume equal to 60. For Form B, the equation to solve is \((10 - 2x)(14 - 2x)x = 48\), or \((5 - x)(7 - x)x = 12\), and one solution is \( x = 3 \).

Grading: +3 points for setting up the equation \( V = 60 \) or \( V = 48 \) and dividing through by 4, +2 points for finding \( x \).
5. [10 pts] Find the equation of the third degree polynomial whose graph is shown below. You may leave your answer in factored form.

Solution: Form A: The polynomial \( p(x) \) has \(-1\) and \(2\) as zeros, but more importantly, the multiplicity of \(-1\) is two, since the curve does not cross the \(x\)-axis at \(x = -1\). Thus, \( p(x) \) is of the form
\[
p(x) = a(x - (-1))^2(x - 2) = a(x + 1)^2(x - 2),
\]
for some value of \(a\). To find this value, note that the point \((0, -\frac{1}{2})\) is a point on the graph of \(p(x)\), so \(p(0) = -\frac{1}{2}\). Substituting 0 for \(x\), we have
\[
-\frac{1}{2} = p(0) = a(0 + 1)^2(0 - 2) = -2a,
\]
or \(a = \frac{1}{4}\).

Hence \(p(x) = \frac{1}{4}(x + 1)^2(x - 2)\).

The idea is similar for Form B. The zeros of \(p(x)\) are \(-1\) and \(2\) (this time with \(2\) having multiplicity two), so
\[
p(x) = a(x + 1)(x - 2)^2,
\]
for some value of \(a\). Using the fact that \(p(0) = 1\), we have
\[
1 = p(0) = a(0 + 1)(0 - 2)^2 = 4a,
\]
or \(a = \frac{1}{4}\),

and so \(p(x) = \frac{1}{4}(x + 1)(x - 2)^2\).

Grading: +5 points for the general form (or finding the roots of \(p(x)\)), +4 points for finding \(a\), and +1 point for rewriting \(p(x)\). Grading for common mistakes: +5 points (total) for fitting the polynomial to a quadratic function; +3 points (total) for trying to fit the polynomial to a quadratic function.
6. [10 pts] Divide the polynomial \( \left\{ \frac{x^3 + 2x^2 - 5}{x^3 + 2x + 5} \right\} \) by \( \left\{ \frac{3x + 1}{2x + 3} \right\} \).

Solution: This problem can be done with synthetic division or long division. The long division for Forms A and B is shown below:

\[
\begin{array}{cccccc}
  & \frac{1}{3} & x^2 & + & \frac{5}{9} x & - & \frac{5}{27} \\
3x + 1 & \underline{\left\{ \right.} & x^3 & + & 2x^2 & - & 5 \\
  & \frac{1}{3} x^3 & + & \frac{1}{3} x^2 & & \\
  & \frac{5}{3} x^2 & & & - & \frac{5}{9} x & + & \frac{5}{9} x & - & \frac{5}{27} \\
  & \frac{5}{9} x & - & \frac{5}{27} & & \\
  & - & \frac{130}{27} & & & \\
\end{array}
\]

\[
\begin{array}{cccccc}
  & \frac{1}{2} & x^2 & - & \frac{3}{4} x & + & \frac{17}{8} \\
2x + 3 & \underline{\left\{ \right.} & x^3 & + & 2x & - & 5 \\
  & \frac{1}{2} x^3 & + & \frac{3}{2} x^2 & & \\
  & \frac{3}{2} x^2 & + & 2x & - & \frac{9}{4} x & - & \frac{9}{4} x & - & \frac{9}{4} x & + & \frac{51}{8} x & + & \frac{17}{8} \\
  & \frac{3}{2} x^2 & + & \frac{17}{8} x & + & \frac{17}{8} x & + & \frac{17}{8} x & + & \frac{17}{8} x & + & \frac{17}{8} x & - & \frac{11}{8} \\
  & - & \frac{11}{8} & & & \\
\end{array}
\]

To use synthetic division to divide \( x^3 + 2x^2 - 5 \) by \( 3x + 1 \), first \( x^3 + 2x^2 - 5 \) must be divided by \( \frac{3x + 1}{3} = x + \frac{1}{3} = x - \left( -\frac{1}{3} \right) \), and then the quotient must be divided by 3. For Form B, \( x^3 + 2x + 5 \) must be divided by \( x + \frac{3}{2} = x - \left( -\frac{3}{2} \right) \), and the quotient then divided by 2. The synthetic divisions are:

\[
\begin{array}{cccc}
  & 1 & 2 & 0 & -5 \\
-1 & -1 & 5 & 5 & -130 \\
3 & 1 & 5 & -9 & -27 \\
\frac{1}{3} x^2 + \frac{5}{3} x - \frac{5}{9} & = \frac{1}{3} x^2 + \frac{5}{9} x - \frac{5}{27} \\
\end{array}
\]

\[
\begin{array}{cccc}
  & 1 & 0 & 2 & 5 \\
-3 & 3 & 9 & 51 & -3 & 17 & 11 \\
2 & -2 & 4 & -8 & -2 & 4 & -8 \\
\frac{x^2 - \frac{3}{2} x + \frac{17}{4}}{2} & = \frac{1}{2} x^2 - \frac{3}{4} x + \frac{17}{8} \\
\end{array}
\]

In either case, the answer is \( \frac{1}{3} x^2 + \frac{5}{9} x - \frac{5}{27} \) with remainder \( -\frac{130}{27} \) for Form A, and \( \frac{1}{2} x^2 - \frac{3}{4} x + \frac{17}{8} \) with remainder \( -\frac{11}{8} \) for Form B.

Grading: For synthetic division, +7 points were allotted to the synthetic division itself, with +3 points for dividing the quotient by 2; Partial Credit: +5 points total if the division was mis-aligned; −2 points if \( -\frac{2}{3} \) was used instead of \( -\frac{3}{2} \); +6 points total for an incomplete division. For long division, Partial Credit was: +3 points total if the division was incomplete, and the quotient started with \( x^2 + \cdots \); +5 points total if the division was mis-aligned, and the quotient started with \( x^2 + \cdots \); +5 points total if fractions were not used (they needed to be here); +7 points total if the quotient started with \( x^2 + \cdots \). +2 points total were given for “complex” division; +5 points were awarded if a “hybrid” approach was attempted.
7. [10 pts] The following is the graph of $f(x)$. Sketch the graph of the function $\{3 - f(x - 2)\}$.

**Solution:** To find the graph of $3 - f(x - 2)$, given the graph of $f(x)$, the graph needs to be transformed in steps. First, the graph needs to be moved to the right 2 units, to get the graph of $f(x - 2)$; then, the graph needs to be reflected about the $x$-axis, to get the graph of $-f(x - 2)$; finally, the graph needs to be moved up 3 units, to get the graph of $3 - f(x - 2)$. Note that the reflection needs to be done before moving the graph up; otherwise, the graph of $-(3 + f(x - 2))$ results.

For Form B, the graph needs to be moved to the right 1 unit, to get the graph of $f(x - 1)$; then it needs to be reflected about the $x$-axis, to get the graph of $-f(x - 1)$; then it needs to be moved up 2 units, to get the graph of $2 - f(x - 1)$. The resulting graphs, for both forms, are shown below:

Grading was done on a 3–7–10 basis; 3 points were awarded if some knowledge of transformations was shown, 7 points if knowledge shown but the steps were done in the wrong order or incorrectly.
8. Let \( f(x) = \frac{3-x}{1+x} \) and \( g(x) = \frac{1-x^2}{x^2+1} \). Find \((f + g)(\frac{1}{2})\).

Answer: Form A: \((f + g)(1) = f(1) + g(1) = (3 - 1) + (1 - 1)^2 = 2\), answer (c); Form B: \((f + g)(2) = f(2) + g(2) = (1 + 2) + (2^2 + 1) = 8\), answer (c).

9. Let \( f(x) = \left\{ \frac{2+3x}{3-x} \right\} \) and \( g(x) = \left\{ \frac{x^2+1}{x^2+x} \right\} \). Find \((f \circ g)(x)\).

Answer: Form A: \((f \circ g)(x) = f(g(x)) = 2 + 3(x^2 + 1) = 3x^2 + 5\), answer (a); for Form B, \((f \circ g)(x) = f(g(x)) = 3 - (x^2 + x) = -x^2 - x - 3\), answer (b).

10. The height of a baseball in feet, \( t \) seconds after being hit, is given by the function

\[
f(t) = -16t^2 + \left\{ \frac{96}{144} \right\} t + 3.
\]

At what time is the baseball at its maximum height?

Answer: Since \( f(x) \) is a quadratic function, its maximum occurs at its vertex. The vertex of \( ax^2 + bx + c = -\frac{b}{2a} \), which can also be found by completing the square: \(-16t^2 + 96t + 3 = -16(t - 3)^2 + 147\), answer (a); \(-16t^2 + 144t + 3 = -16(t - 4.5)^2 + 327\), answer (c).

11. What is the left- and right-hand behavior of the polynomial \( p(x) = \left\{ \frac{x^5 + 16x^4 - 8x - 5}{2x^4 - 3x^3 + x^5 - 2} \right\} \)?

Answer: The left- and right-hand behavior of a polynomial are the same as the left- and right-hand behavior of the term with the largest power of \( x \). For Form A, this is the \( x^5 \) term. For Form B, the term which determines the left- and right-hand behavior is \( x^5 \), which falls to the left and rises to the right, answer (c).

12. Let \( q(x) = \left\{ \frac{3x^3 - 2x^2 + 5x - 4}{9x^5 - 4x^4 + 3x - 2} \right\} \). Use the Rational Root Test to list all potential rational roots of \( q(x) \).

Answer: Form A: The potential rational roots of \( q(x) \) are an integer which divides evenly into \(-4\) divided by an integer which divides evenly into \(3\); all rational numbers of this form are listed in (c). Form B: The potential rational roots of \( q(x) \) are an integer which divides evenly into \(-2\) divided by an integer which divides evenly into \(9\), answer (d).

13. What is the multiplicity of the root \( x = 0 \) in the polynomial \( \left\{ \frac{x^5 - 2x^4 + x^2}{x^8 - x^4 + x} \right\} \)?

Answer: The multiplicity of \( x = 0 \) in \( p(x) \) is the largest power \( n \) so that \((x - 0)^n \) divides evenly into \( p(x) \). For Form A, \( x^5 - 2x^4 + x^2 = x^2(x^3 - 2x + 1) \); so \((x - 0)^2 = x^2 \) divides evenly into \( p(x) \), but \( x^3 \) doesn’t, so the answer is (b). For Form B, \( x^8 - x^4 + x = x^1(x^7 - x^3 + 1) \), so the multiplicity of \( x = 0 \) in \( p(x) \) is 1, answer (a).