Solutions to MAT 117 Test #1

Because there are two versions of the test, solutions will only be given for Form A. Differences from the Form B version will be given. (The values for Form A appear above those of Form B in the curly braces {}.)

1. Consider the graph of the equation \[
\begin{align*}
  x^2 + xy + y &= 1 \\
  xy - x + 2y^2 &= 3
\end{align*}
\].

a. **[6 pts] Algebraically** find the \(x\)-intercept(s) of the graph of the equation.

*Solution:* The \(x\)-intercepts of a graph are the points where \(y = 0\). Substituting 0 for \(y\), we get the equation \(x^2 = 1\), so \(x = 1\) and \(x = -1\) [for Form A]. For Form B, the procedure is the same, except you get \(-x = 3\), so \(x = -3\).

*Grading:* +3 points for setting \(y = 0\); +3 points for finding \(x\). −2 points for forgetting \(x = -1\) in Form A.

b. **[6 pts] Algebraically** find the \(y\)-intercept(s) of the graph of the equation.

*Solution:* The procedure is the same as in (a), except now \(x = 0\). For Form A, you get \(y = 1\), and for Form B, the equation is \(2y^2 = 3\), or \(y = \sqrt{\frac{3}{2}}\) or \(y = -\sqrt{\frac{3}{2}}\).

*Grading:* Same as (a).

2. **[10 pts]** Find the equation in standard form of the circle whose equation is given by

\[
\begin{align*}
  x^2 - 8x + y^2 + 4y &= 16 \\
  x^2 + 6x + y^2 - 8y &= 24
\end{align*}
\]

Also find the center of this circle and its radius.

*Solution:* The standard form of the equation of a circle is \((x - h)^2 + (y - k)^2 = r^2\), where \((h, k)\) is the center of the circle, and \(r\) is the radius.

To put the given equation into standard form, you need to complete the square. For Form A, this involves adding \(\left(\frac{-8}{2}\right)^2 = 16\) to both sides, to get

\[
(x^2 - 8x + 16) + (y^2 + 4y) = 16 + 16,
\]

and then you need to add \(\left(\frac{4}{2}\right)^2 = 4\) to both sides of the equation, to get

\[
(x^2 - 8x + 16) + (y^2 + 4y + 4) = 16 + 16 + 4.
\]
The expressions inside the parentheses factor into perfect squares. If you do this and add the numbers on the right-hand side, you get

\[(x - 4)^2 + (y + 2)^2 = 36 = 6^2.\]

The center of the circle is thus \((4, -2)\) and the radius is 6. For Form B, you need to complete the square for the equation

\[(x^2 + 6x) + (y^2 - 8y) = 24.\]

You do this by adding \(\left(\frac{6}{2}\right)^2 = 9\) to both sides of the equation, and then adding \(\left(-\frac{8}{2}\right)^2 = 16\) to both sides of the equation. You then get

\[(x^2 + 6x + 9) + (y^2 - 8y + 16) = 24 + 9 + 16.\]

Factoring the expressions in parentheses, you get

\[(x + 3)^2 + (y - 4)^2 = 49 = 7^2,\]

and the center of this circle is \((-3, 4)\), and its radius is 7.

Grading: +2 points for adding the numbers inside the parentheses (twice), +3 points for factoring and simplifying, +3 points for finding the center and radius. Partial credit: +3 points for writing down the standard form of the equation of a circle.

3. [10 pts] Find an equation of the line that passes through the point \(\{(2, -2), (-2, 2)\}\) and is parallel to the line \(\{3x + 5y = 8, 5x - 3y = 2\}\). Write this equation in slope-intercept form.

Solution: To find an equation for such a line, you need to find a point on that line (which is given to you) and its slope. Since the desired line should be parallel to another given line, the slopes should be the same.

For Form A, the equation for the other line is \(3x + 5y = 8\). To obtain the slope, solve for \(y\):

\[5y = 8 - 3x, \text{ or } y = \frac{8}{5} - \frac{3}{5}x,\]

so the slope of the desired line is \(-\frac{3}{5}\). We put this slope and the point \((2, -2)\) into the point-slope form:

\[y - (-2) = -\frac{3}{5}(x - 2).\]

To get this equation into slope-intercept form, we solve for \(y\), getting

\[y = -\frac{3}{5}(x - 2) - 2 = -\frac{3}{5}x + \frac{6}{5} - 2 = -\frac{3}{5}x - \frac{4}{5}.\]

For Form B, the procedure is the same, except the slope of the other line is \(\frac{5}{3}\) and the line goes through the point \((-2, 2)\). The point-slope form of the equation of the line in question is

\[y - 2 = \frac{5}{3}(x + 2),\]
and the point-intercept form can be found by solving for $y$; the answer is
\[ y = \frac{5}{3}x + \frac{16}{3}. \]

Grading: +3 points for the point-slope form of the equation of a line; +2 points for finding the slope of the given line, +2 points for substituting into the point-slope form; +3 points for finding the slope-intercept form of the equation. Grading for common errors: −7 points for giving the equation of the other line.

4. [6 pts] Does the equation \( \begin{align*}
    x^2 + 2xy &= y - 1 \\
x + 2xy &= y^2 - 1
\end{align*} \) represent a function? Be sure to show your work.

Solution: To determine whether the equation represents a function, you need to check that for any value of $x$, there is at most one value for $y$ which goes along with it. To do this algebraically, you solve for $y$. For Form A, this involves the following steps:

\[
\begin{align*}
    x^2 + 2xy &= y - 1 \\
x^2 + 1 &= y - 2xy = y(1 - 2x) \\
\frac{x^2 + 1}{1 - 2x} &= y.
\end{align*}
\]

Since the final equation gives at most one value of $y$ for every value of $x$, the equation does represent a function.

For Form B, we do the manipulation:

\[
\begin{align*}
x + 2xy &= y^2 - 1 \\
0 &= y^2 - 2xy - 1 - x \\
y &= \frac{+2x \pm \sqrt{4x^2 + 4(1 + x)}}{2}.
\end{align*}
\]

Since in the final row, more than one value of $y$ goes along with $x = 0$, this equation does not represent a function.

Grading: −3 points for the wrong answer (with work); −5 points for the wrong answer (without work); −3 points for the correct answer without enough work.
5. A contractor purchases a piece of equipment for \( \{ \$40,600 \ \$35,200 \} \). The equipment requires an expenditure of \( \{ \$7 \ \$8 \} \) per hour for fuel and maintainence, and the operator is paid \( \{ \$10.75 \ \$11.50 \} \) per hour.

a. [4 pts] Write a linear equation giving the total cost \( C \) of operating this equipment for \( x \) hours. Be sure to include the price of the equipment.

Solution: The cost of operating the equipment includes the original price of the equipment, plus the cost needed for fuel and maintainence, plus the cost of paying the operator. The total cost is then

\[ C = 40,600 + 7x + 10.75x = 17.75x + 40,600, \]

for Form A; for Form B, it is

\[ C = 35,200 + 8x + 11.50x = 19.50x + 35,200. \]

Grading for common mistakes: -2 points if any term is missing.

b. [4 pts] Customers are charged \( \{ \$35 \ \$40 \} \) per hour of machine use. Write a linear equation for the revenue \( R \) that the contractor receives from \( x \) hours of use.

Solution: The revenue is the amount of money paid to the contractor; for Form A, this is \( R = 35x \), and for Form B, it is \( R = 40x \).

Grading for common errors: -2 points for giving the profit, not revenue.

c. [4 pts] Find the number of hours this equipment needs to be used to reach the break-even point.

Solution: The break-even point is where \( C = R \). For Form A, the value of \( x \) that makes \( C = R \) is

\[ 17.75x + 40,600 = 35x \]
\[ 40,600 = 17.25x \]
\[ x = \frac{40,600}{17.25} \approx 2353.6 \text{ hours}, \]

and for Form B, the answer is

\[ 35,200 + 19.50x = 40x \]
\[ 35,200 = 20.50x \]
\[ x = \frac{35,200}{20.50} \approx 1717.1 \text{ hours}. \]

Grading: +2 points for stating that \( C = R \); +2 points for finding \( x \). Grading for common errors: +2 points (total) for \( \frac{35,200}{40} \) or \( \frac{40,600}{35} \).
6. Find the domains of the following functions:

a. [6 pts] \( f(x) = \begin{cases} \sqrt{x + 1} \\ \frac{x^2 + 1}{x^2 - 9} \end{cases} \)

Solution: The function is given algebraically, so those rules for domains have to be used: You are not allowed to divide by zero, and you are not allowed to take the square root of a negative number. For Form A, this means that the domain consists of all values of \( x \) such that \( x + 1 \geq 0 \) and \( x^2 + 1 \neq 0 \). Since \( x^2 + 1 \geq 1 \) for all real values of \( x \), the second condition does not rule out any values of \( x \). The first one indicates that \( x \geq -1 \), so the domain of the function \( f(x) \) is all real numbers which are at least \(-1\), or, in interval notation, \([-1, +\infty)\).

For Form B, the conditions for \( x \) to be in the domain are that \( x - 1 \geq 0 \) and \( x^2 - 9 \neq 0 \). The second equation says that \( x \neq 3, -3 \), and the first condition is that \( x \geq 1 \). The answer is then “all real numbers at least 1, except 3”, or, in interval notation, \([1, 3) \cup (3, +\infty)\).

Grading: +2 points for the square root rule; +2 points for the division rule; +2 points for finding the domain. Grading for common mistakes: –1 point for omitting –1 (Form A) or 1 (Form B); –2 points for saying “\( x > -2 \)” (Form A); –4 points for “all reals”; –3 points for “all reals except for –1” (Form A) or “all reals except for 1” (Form B).

b. [6 pts] \( g(x) \), given by the following graph:

Solution: The function \( g(x) \) is given graphically. To find the domain of a function when its graph is given, drop or raise all points in the graph onto the \( x \)-axis. In Form A, one piece of the graph (the curve between \((-2, -1)\) and \((-1, -2)\)) produces the interval \((-2, -1]\) and the other one (the curve between \((-1, 1)\) and \((4, 3)\)) the interval \((-1, 4]\). These intervals make up the interval \((-2, 4]\), or “all real numbers between \(-2\) and \(4\), including \(4\) but excluding \(-2\).” For Form B, the two curves produce the interval \([-3, 1)\) (from the curve between \((-3, 1)\) and \((1, -1)\)) and the interval \((1, 4)\) (from the curve between \((1, 3)\) and \((4, 0)\)). The domain of \( g(x) \) is then \([-3, 1) \cup (1, 4]\), or “all real numbers between \(-3\) and \(4\) (including \(-3\) and \(4\)) except for 1.”
Grading for common mistakes: −1 point for a wrong endpoint; −2 points for including 1 in the domain (Form B); +2 points (total) for saying where the function is increasing/decreasing; +2 points (total) for giving the range (−2, −1) ∪ [−0.5, 3] for Form A, (−1, 3) for Form B.

7. [8 pts] A small college had \( \{2732, 2288\} \) students in \( \{1997, 1998\} \) and \( \{2846, 2473\} \) in \( \{1999, 2000\} \). If the enrollment follows a linear growth pattern, how many students will the college have in \( \{2006, 2004\} \)?

Solution: The enrollment of this problem follows a linear growth model, so \( y = mx + b \), where \( y \) is the number of students in the college, \( x \) represents the year, and \( m \) and \( b \) are constants which need to be determined. For convenience, \( x \) will represent the number of years since 1990, so \( x = 7 \) will represent the year 1997.

The slope \( m \) is given by the rate of increase; for Form A, this is

\[
m = \frac{2846 - 2732}{9 - 7} = 57;
\]

for Form B, it is

\[
m = \frac{2473 - 2288}{10 - 8} = 92.5.
\]

Now you need to find the value of \( b \). For Form A, when \( x = 7, y = 2732 \), so

\[
2732 = 57 \cdot 7 + b, \quad \text{or} \quad b = 2732 - 57 \cdot 7 = 2333,
\]

and the model indicates that \( y = 57x + 2333 \). For Form B, we have \( y = 2288 \) when \( x = 8 \), so

\[
2288 = 92.5 \cdot 8 + b, \quad \text{or} \quad b = 2288 - 92.5 \cdot 8 = 1548,
\]

and the model indicates that \( y = 92.5x + 1548 \).

Now, to find the expected enrollment in 2006 (for Form A), we find \( y \) when \( x = 16 \): \( y = 57 \cdot 16 + 2333 = 3245 \); for Form B, the answer is when \( x = 14 \): \( y = 92.5 \cdot 14 + 1548 = 2843 \).

Grading: +2 points for setting up the model, +2 points for finding \( m \), +2 points for finding \( b \), +2 points for finding the final answer.
8. \{ 
\text{Find the distance between the points \((4, -4)\) and \((-1, 8)\).} 
\text{Find the distance between the points \((-2, 2)\) and \((6, -4)\).} 
\}

\text{Answer: Use the distance formula; the distance is } \sqrt{(4 - (-1))^2 + (-4 - 8)^2} = \sqrt{169} = 13, \text{ for Form A; } \sqrt{(6 - (-2))^2 + (-4 - 2)^2} = \sqrt{100} = 10 \text{ for Form B.} \]

9. \{ 
\text{Find the midpoint of the line segment joining the points \((2, -4)\) and \((4, 6)\).} 
\text{Find the point halfway between the points \((1, 8)\) and \((7, -2)\).} 
\}

\text{Answer: Use the midpoint formula: For Form A, the answer is } \left( \frac{2 + 4}{2}, \frac{-4 + 6}{2} \right) = (3, 1); \text{ for Form B,} \left( \frac{1 + 7}{2}, \frac{8 + (-2)}{2} \right) = (4, 3). \]

10. \{ 
\text{If } f(x) = x^3 - 2x^2 + 3, \text{ what is } f(-2)? \}
\text{If } f(x) = 2x^2 - 3x + 6, \text{ what is } f(-1)? 
\}

\text{Answer: Form A: } (-2)^3 - 2(-2)^2 + 3 = -13; \text{ Form B: } 2(-1)^2 - 3(-1) + 6 = 11. \]

11. \{ 
\text{Given that } g(x) = \frac{3 + 2x}{4 - x}, \text{ find } g(a - 4) \text{ and simplify.} 
\text{Given that } g(x) = \frac{2 + x}{1 - 2x}, \text{ find } g(a - 1) \text{ and simplify.} 
\}

\text{Answer: Form A: } \frac{3 + 2(a - 4)}{4 - (a - 4)} = \frac{3 + 2a - 8}{4 - a + 4} = \frac{2a - 5}{8 - a}. \text{ Form B: } \frac{2 + (a - 1)}{1 - 2(a - 1)} = \frac{2 + a - 1}{1 - 2a + 2} = \frac{a + 1}{3 - 2a}. \]

12. Determine whether the function \(h(x) = \begin{cases} 
\frac{x^3}{1 - 2x^2} \\
\frac{x^2 + x - 1}{x^2 + |x|} 
\end{cases}\) is even, odd, or neither.

\text{Answer: Form A: } h(-x) = \frac{(-x)^3}{1 - 2(-x)^2} = -\frac{x^3}{1 - 2x^2} = -h(x), \text{ so } h(x) \text{ is odd; Form B: } h(-x) = (-x)^2 + (-x) - 1 = x^2 - x - 1, \text{ and this is not } h(x) \text{ or } -h(x), \text{ so } h(x) \text{ is neither odd or even.} \]

13. Which of the following lines is perpendicular to the line whose equation is given by \( \begin{cases} 
9x + 6y = 1 \\
-4x + 6y = 15 
\end{cases} \)?

\text{Answer: Form A: The slope of the line given by } 9x + 6y = 1 \text{ is } -\frac{3}{2}, \text{ so is perpendicular to a line with slope } \frac{2}{3}. \text{ The line in (c) has slope } \frac{5 - 1}{1 - (-5)} = \frac{2}{3}. \text{ Form B: The line with the equation } -4x + 6y = 15 \text{ has slope } \frac{2}{3}, \text{ and is perpendicular to a line with slope } -\frac{3}{2}. \]

MAT 117 Website: \url{http://math.asu.edu/~checkman/117/}