MATH 342 Test #1

The following provides detailed solutions to version C of the test, along with the answers to version F (the process is the same), as well as the grading scheme which was used.

This was a test which could have required a lot of computation to complete. If you knew several ways to solve the problem, you should have chosen the one that took the least amount of time/work. (And, yes, I did write up the keys for both tests in a combined time of 40 minutes.)

(1) Find all solutions to the following system of linear equations.

\[
\begin{align*}
 x_1 - 3x_2 + 7x_3 + 2x_4 &= 4 \\
 x_1 - x_2 + 3x_3 + 2x_4 &= 1 \\
 3x_1 - 7x_2 + 17x_3 + 8x_4 &= 10 \\
 x_1 + x_3 - x_4 &= -2
\end{align*}
\]

**Solution:** Putting the system of linear equations into augmented matrix form and performing row operations, we have [for version C of the test]:

\[
\begin{align*}
\begin{bmatrix}
1 & -3 & 7 & 2 & 4 \\
1 & -1 & 3 & 2 & 1 \\
3 & -7 & 17 & 8 & 10 \\
1 & 0 & 1 & -1 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & -1 & 0 & 2 & -4 \\
0 & 2 & -4 & 2 & -2 \\
0 & 3 & -6 & -3 & -6 \\
0 & 1 & 2 & -4 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & 7 & 2 & 4 \\
0 & 1 & -2 & 1 & -1 \\
0 & 0 & -2 & -1 & -1 \\
0 & 0 & 0 & -6 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & 0 & -\frac{3}{2} \\
0 & 1 & -2 & 0 & -\frac{3}{2} \\
0 & 0 & 0 & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

Since not every column of this matrix has a pivot, the system has more than one solution. So, to solve it, we must find the row-reduced form of the matrix. **Continuing:**
Since there is no pivot in the third column, the third variable \( x_3 \) is a free variable. The rest of the variables have the values
\[
\begin{align*}
  x_1 &= -\frac{3}{2} - x_3 \\
  x_2 &= -\frac{3}{2} + 2x_3 \\
  x_4 &= \frac{1}{2}
\end{align*}
\]
Row reduction is used to solve problem (5) of the F version of the test. The answer turns out to be
\[
\begin{align*}
  x_1 &= \frac{1}{2} + x_3 \\
  x_2 &= -\frac{3}{2} - 2x_3 \\
  x_4 &= -\frac{3}{2}.
\end{align*}
\]
Grading: +10 points for the row reduction, +10 points for interpreting the result. If Cramer’s Rule was used instead, +10 points were given for the formula, and +5 points for evaluating \( \det A = 0 \). −1 points for arithmetic, −2 points for not mentioning that \( x_3 \) is a free variable.

(2) Suppose that \( A^{-1} = A^3 \) (or \( A^7 \)). What can you say about \( \det A \)?

Solution: Taking the determinant of both sides of this equation, we get
\[
\det A^{-1} = \det A^3,
\]
or
\[
\frac{1}{\det A} = (\det A)^3,
\]
which is equivalent to
\[
(\det A)^4 = 1.
\]
Since the only real numbers which satisfy this are 1 and −1, the determinant of \( A \) must be one of these values.

In problem (4) of version F, taking the determinant of both sides of the equation yields
\[
\frac{1}{\det A} = (\det A)^7,
\]
which also leads to \( \det A = \pm 1 \).

Grading: +10 points for taking the determinant of both sides of the equation, +10 points for \( \det A = \pm 1 \). +18 points total for \( \det A = 1 \); +17 points for \( \det A = 0 \) or 1; +5 points for nothing (is learned about \( \det A \)); +10 points for \( \det A \neq 0 \); +5 points for \( \det A = 0 \) (this is not true, because \( A \) is invertible); +5 points for \( \det A \) being a matrix divided by another matrix; +5 points for substantial work.
(3) Find the (3, 1) entry of $A^{-1}$, where

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 4 \\
0 & -1 & 1 & 3 & 5 \\
0 & 0 & 2 & 4 & 7 \\
0 & 0 & 0 & -2 & 9 \\
2 & 0 & 0 & 0 & 3
\end{pmatrix}$$

Solution: There are a couple ways to solve this one. The method that was intended was Cramer’s Rule, which says that

$$(A^{-1})_{3,1} = \frac{C_{1,3}}{\det A} = \frac{(-1)^{1+3} \det M_{1,3}}{\det A},$$

where $C_{1,3}$ is the (1, 3) cofactor of $A$ and $M_{1,3}$ is the (1, 3) minor of $A$. The determinants of these two matrices need to be calculated. Then

$$\det A = \begin{vmatrix}
1 & 0 & 0 & 0 & 4 \\
0 & -1 & 1 & 3 & 5 \\
0 & 0 & 2 & 4 & 7 \\
0 & 0 & 0 & -2 & 9 \\
2 & 0 & 0 & 0 & 3
\end{vmatrix} = (1)(-1)(2)(-5) = -20, \text{ and}$$

$$\det M_{1,3} = \begin{vmatrix}
0 & -1 & 3 & 5 \\
0 & 0 & 4 & 7 \\
0 & 0 & -2 & 9 \\
2 & 0 & 0 & 3
\end{vmatrix} = (-1)^{1+4} \cdot 2 \cdot \begin{vmatrix}
-1 & 3 & 5 \\
0 & 4 & 7 \\
0 & -2 & 9
\end{vmatrix}, \text{ by expansion by minors along the first column,}$$

$$= (-2) \cdot (-1)^{1+1} \cdot (-1) \cdot \begin{vmatrix}
4 & 7 \\
-2 & 9
\end{vmatrix}, \text{ by expansion by minors along the first column,}$$

$$= 2 \cdot (4 \cdot 9 - (-2) \cdot 7) = 100.$$
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & -1 & 2 & 3 & 0 \\
0 & 0 & -2 & -1 & 3 \\
0 & 0 & 0 & 2 & -2 \\
3 & 0 & 0 & 0 & 4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and now there are no further row operations necessary. The first four rows of \( A^{-1} \) still need to be calculated, but the fifth row will remain as \( \begin{pmatrix} 3/2, 0, 0, 0, -1/2 \end{pmatrix} \), and so the \( (5,1) \) entry will be \( 3/2 \).

Grading: +10 points for the formula for Cramer’s Rule, +5 points for each determinant; or +10 points for setting up the row reduction and +10 points for doing it. –1 point for arithmetic errors, –2 points for finding the \( (3,1) \) minor of \( A \) (for version C) or the \( (5,1) \) minor of \( A \) (version F) (the indices need to be swapped).

(4) Solve for the square matrix \( X \) in the following equation. Assume that \( A, B, \) and \( C \) are square matrices and that all relevant matrices are invertible.

version C: \[ AX + B = X + C \]
version F, (2): \[ XA + B = X + C \]

Solution: The idea is to solve for \( X \), but care needs to be taken once the equation looks like \( MX = N \) (or \( XM = N \)); the division had to be done on the correct side to receive full credit. (Matrix multiplication is not commutative, in general.)

For version C, the critical equation is \( AX - X = C - B \), obtained by subtracting \( X \) and \( B \) from both sides of the initial equation. Now an \( X \) needs to be factored out, so we need to rewrite \( AX - X \) as \( AX - IX \), so we get \( (A - I)X = C - B \). Now comes the critical division/multiplication: We need to multiply on the left by \( (A - I)^{-1} \) to solve for \( X \). We also need to be sure that we multiply on the left on the right-hand side. This gives us

\[ X = (A - I)^{-1}(C - B). \]
The solution for version F is similar, except that \((A - I)^{-1}\) needs to be multiplied on the right side of each side of the equation. The solution is then

\[ X = (C - B)(A - I)^{-1}. \]

Grading: +10 points for the equation \((A - I)X = C - B\) (or \(X(A - I) = C - B\)), and +10 points for dividing on the correct side. −5 points for \((A - I)^{-1}(C - B)\) when \((C - B)(A - I)^{-1}\) was the correct answer (and vice-versa); −5 points for the ambiguous expression \(\frac{C - B}{A - I}\), −5 points for \(X = A^{-1}(X + C - B)\) or \((X + C - B)A^{-1}\); −2 points for \(B - C\) instead of \(C - B\); −1 point for \(A + I\) being where \(A - I\) should have been.

(5) Find a matrix \(E\) such that \(EA = B\), where \(A\) and \(B\) are the matrices shown below.

\[
A = \begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 2 & 1 & 3 \\
-2 & 0 & 6 & 4 \\
-3 & 1 & 4 & -1
\end{pmatrix} \quad B = \begin{pmatrix}
-3 & 1 & 4 & -1 \\
0 & 2 & 1 & 3 \\
1 & 0 & -3 & -2 \\
1 & -1 & 0 & 1
\end{pmatrix}
\]

Solution: Like problem (3) ((1) in version F), there are a couple of ways to do this. One way was to solve for \(E\), obtaining the equation \(E = BA^{-1}\), finding \(A^{-1}\), and then multiplying it on the left by \(B\). This method takes a bit of computation, but full credit was awarded for completing the problem this way. (Note that if \(A\) had turned out not to be invertible, then this method would not work. I can’t recall whether I checked to see whether \(A\) was invertible or not, so this was likely a “happy accident.”) Since \(A\) is invertible, this also implies that \(E\) is unique; that there is only one such matrix \(E\).

The quicker way of doing this problem is recalling that this problem is similar to a problem in the book, where a matrix \(E\) turned out to be an elementary matrix if exactly one row operation was required to convert \(A\) into \(B\). Here, two row operations are necessary, so the elementary matrices needed to be multiplied together.

Comparing the rows of \(A\) and \(B\), it is observed that row 1 of \(A\) is the same as row 4 of \(B\), row 2 of \(A\) is the same as row 2 of \(B\), and row 4 of \(A\) is the same as row 1 of \(B\). Hence it looks like rows 1 and 4 have been swapped, and indeed this is the case. Comparing the third rows of \(A\) and \(B\), we see that we can obtain the third row of \(B\) from the third row of \(A\) by dividing it by −2.

The elementary matrix \(E_1\) for swapping rows 1 and 4 (of a matrix with 4 rows) is

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]
and the elementary matrix $E_2$ for dividing the third row by $-2$ is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix};
\]
the easy way to remember these forms is to do the appropriate row operation on the identity matrix.

Thus $E_1A$ is the matrix $A$ after swapping rows 1 and 4, and $E_2E_1A$ is that matrix after dividing the third row by $-2$. Since $E_2E_1A = B$, we can let $E = E_2E_1$.

Computation shows that
\[
E = E_2E_1 = 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot 
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]
(The matrices $E_1$ and $E_2$ commute, since you can do the row operations in either order and end up with the same matrix $B$, so $E = E_1E_2$ as well.)

For problem (3) in version F of the test, the row operations needed to convert $A$ to $B$ are swapping rows 1 and 3, and then dividing the second row by $-3$. The elementary matrices $E'_1$ and $E'_2$ are constructed as above, and the answer is
\[
E' = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & -\frac{1}{3} & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Lastly, one could solve this problem by writing
\[
E = \begin{pmatrix}
e_1 & e_2 & e_3 & e_4 \\
e_5 & e_6 & e_7 & e_8 \\
e_9 & e_{10} & e_{11} & e_{12} \\
e_{13} & e_{14} & e_{15} & e_{16}
\end{pmatrix}
\]
and then writing the equation $EA = B$ as a system of linear equations with 16 variables and 16 equations and then solving for $e_1, e_2, \ldots, e_{16}$. The time limit prohibited one from completing this, however.

Grading: USING ROW OPERATIONS: +10 points for finding the two row operations, +5 points for finding $E_1$ and $E_2$, +5 points for finding $E$; -1 points for arithmetic errors, -3 points for copying row 2 (row 4 for version F), -2 points for multiplying the third (resp. second) row by $-2$ (resp $-3$) instead of dividing by it.

USING $BA^{-1}$: +10 points for solving for $E$, +5 points for finding $A^{-1}$, +5 points for finding $E$; -5 points for $E = A^{-1}B$; -2 points for an incomplete $A$ or $E$. Typically, +13 total points were awarded for most of the work.

USING THE SYSTEM OF LINEAR EQUATIONS: 10 points for setting up, +10 points for finding the $e_i$'s. +5 points for substantial work.

MAT 342 Website: http://math.la.asu.edu/~checkman/342/