Solutions to MAT 117 Test #3

Because there are two versions of the test, solutions will only be given for Form C. Differences from the Form D version will be given. (The values for Form C appear above those of Form D in the brackets.)

1. [5 pts] Simplify algebraically:

\[
\begin{bmatrix}
5 - 3i \\
4 - i \\
5 - 9i \\
8 - i
\end{bmatrix}
\]

\[
\text{Solution: Multiply the numerator and denominator by the conjugate of the denominator to "rationalize the denominator":}
\]

\[
\frac{5 - 3i}{4 - i} \cdot \frac{4 + i}{4 + i} = \frac{(5 - 3i)(4 + i)}{(4 - i)(4 + i)} = \frac{20 + 5i - 12i - 3i^2}{16 + 4i - 4i - i^2} = \frac{20 - 7i + 3}{16 + 1} = \frac{23 - 7i}{17}.
\]

For Version D:

\[
\frac{5 - 9i}{8 - i} \cdot \frac{8 + i}{8 + i} = \frac{(5 - 9i)(8 + i)}{(8 - i)(8 + i)} = \frac{40 + 5i - 72i - 9i^2}{64 + 8i - 8i - i^2} = \frac{40 - 67i + 9}{64 + 1} = \frac{49 - 67i}{65}.
\]

Grading: +3 points for multiplying numerator and denominator by 4 + i (or 8 + i), +2 points for simplifying. Grading for common errors: -3 points for subtraction instead of division; -2 points if the numerator and denominator were multiplied by 4 - i or i - 4 (or 8 - i or i - 8); -1 point if the denominator was missing.

2. [5 pts] Find all roots of

\[
\begin{bmatrix}
x^5 - 7x^4 + 18x^3 - 26x^2 + 32x - 24 \\
x^5 - x^4 - 7x^3 - 11x^2 - 8x + 12
\end{bmatrix} = 0
\]

\[
\text{Solution: The Rational Root Test implies (for version C) that every rational solution is one of ±1, ±2, ±3, ±4, ±6, ±8, ±12, ±24. These values can be tested using synthetic division, long division, substitution, or graphing. (A graphing calculator can be used with XMIN = -24, XMAX = 24, YMIN = -20, and YMAX = 20; you are then looking to see where the graph crosses the x-axis.) The numbers 2 and 3 turn out to be roots. Dividing } x^5 - 7x^4 + 18x^3 - 26x^2 + 32x - 24 \text{ by } x - 2 \text{ yields:}
\]

\[
\begin{array}{r|rrrrr}
   & 1 & -7 & 18 & -26 & 32 & -24 \\
\hline
2 & 2 & -10 & 16 & -20 & 24 \\
\hline
   & 1 & -5 & 8 & -10 & 12
\end{array}
\]

or \( x^4 - 5x^3 + 8x^2 - 10x + 12 \). Dividing this quotient by \( x - 3 \) yields:

\[
\begin{array}{r|rrrr}
   & 1 & -5 & 8 & -10 \\
\hline
3 & 3 & -6 & 2 & -12 \\
\hline
   & 1 & -2 & 2 & -4
\end{array}
\]
or $x^3 - 2x^2 + 2x - 4$. This polynomial has one more “nice” root: 2. Dividing it by $x - 2$ produces

\[
\begin{array}{c|cccc}
1 & -2 & 2 & -4 \\
\hline
2 & 2 & 0 & 4 \\
1 & 0 & 2 \\
\end{array}
\]

or $x^2 + 2$. This polynomial has no real roots; the quadratic formula provides the remaining roots: $\pm \sqrt{2}i$. The polynomial then has the following roots: 2 (with multiplicity 2), 3, and $\pm \sqrt{2}i$.

In version D, the Rational Root Test implies that $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$ are the only rational numbers which can be roots; the ones which turn out to be genuine roots are 2 and $-3$. Dividing $x^5 - x^4 - 7x^3 + 11x^2 - 8x + 12$ by $x - 2$ and $x + 3$ produces $x^3 - 2x^2 + x - 2$. This cubic polynomial has 2 as a root, and $x^3 - 2x^2 + x - 2$ divided by $x - 2$ is $x^2 + 1$, which has the roots $\pm i$. The roots of $x^5 - x^4 - 7x^3 + 11x^2 - 8x + 12$ are then 2 (multiplicity 2), $-3$, and $\pm i$. (And if you think this problem was bad, you should check out the polynomial on the standard 117 test . . . )

Grading: +1 point per root, +1 point for doing the Rational Root Test. Grading for common mistakes: −1 point for $\pm \sqrt{2}$ (version C), $\pm 2i$ (version C), $-2, -2, -3$ collectively (version C), $-2, -2, 3$ collectively (version D), $\pm 1$ (version D).

3. **[5 pts]** Find a third degree polynomial with integer coefficients that has the given zeros:

\[
x = 1 \quad x = \left\{ 5i \right\} \left\{ 6i \right\}
\]

**Solution:** If a polynomial with integer coefficients has $5i$ as a root, it must also have its conjugate ($-5i$) as a root. Thus, this polynomial must have $x - 1$, $x - 5i$, and $x + 5i$ as factors. One such polynomial is

\[
(x - 1)(x - 5i)(x + 5i) = (x - 1)(x^2 + 25) = x^3 - x^2 + 25x - 25.
\]

For version D, a polynomial that works is

\[
(x - 1)(x - 6i)(x + 6i) = (x - 1)(x^2 + 36) = x^3 - x^2 + 36x - 36.
\]

(Any integer, nonzero multiple of this polynomial will also work, incidentally.)

Grading: +3 points for the factored form, +2 points for multiplying out. Grading for common mistakes: −1 point for $(x - 1)(x^2 + 5)$ (C) or $(x - 1)(x^2 + 6)$ (D), −2 points if the coefficients were not integers; −3 points for $(x - 1)(x - 6i)$ (D).
4. For the function \( f(x) = \begin{cases} \frac{x}{16x^2 - 9} \\ \frac{x}{9x^2 - 4} \end{cases} \)

   a. [2 pts] Find the \( x \)-intercept(s).

   Solution: The \( x \)-intercept(s) is/are the values of \( x \) such that \( f(x) = 0 \). Since \( f(x) \) is a rational function, \( f(x) = 0 \) when the numerator is zero. Hence the \( x \)-intercept of \( f(x) \) is 0 (for C and D).

   b. [2 pts] Find the \( y \)-intercept(s).

   Solution: The \( y \)-intercept is where \( x = 0 \), or \( f(0) \). Since \( f(0) = 0 \), the \( y \)-intercept (for C and D) is 0.

   c. [2 pts] Find the equation(s) of the horizontal asymptote(s).

   Solution: Whether \( f(x) \) has a horizontal asymptote depends on what the leading terms of the numerator and denominator are; i.e., the long-term behavior. When \( x \) is big, \( f(x) \) is approximately \( \frac{x}{16x^2} = \frac{1}{16x} = 0 \) (for C), or \( \frac{x}{9x^2} = \frac{1}{9x} = 0 \) (for D). In either case, \( f(x) \) has a horizontal asymptote of \( y = 0 \) (since the long-term behavior of \( f(x) \) is 0).

   d. [2 pts] Find the equation(s) of the vertical asymptote(s).

   Solution: The vertical asymptotes of a rational function are the values of \( x \) where you are dividing by zero; they are where the denominator is zero. For (C), they are the solutions to \( 16x^2 - 9 = 0 \) or \( x = \pm 3/4 \); for (D) they are the solutions to \( 9x^2 - 4 = 0 \): \( x = \pm 2/3 \).

   e. [2 pts] Sketch a complete graph, incorporating the information found in parts a–d.

   Solution: See the webpage at http://math.la.asu.edu/~checkman/117/soln3.html.

   Grading: parts (a)–(d) were on a 0-1-2 point basis; for (e), +2 points were awarded only if the correct graph was sketched, or if all the work for (a)–(d) was present, and the graph was based on this work.

5. [5 pts] We invest $\begin{cases} 4,500 \\ 7,500 \end{cases}$ into an account that compounds quarterly at an annual rate of $\begin{cases} 16^{1/2} \\ 18^{1/2} \end{cases}$%. How much is in the account after $\begin{cases} 7^{1/2} \\ 9^{1/2} \end{cases}$ years?

Solution: Use the compound interest formula (provided on the test). For (C), \( P = 4500, r = 16.5\% = 0.165, n = 4 \) (compounded quarterly), and \( t = 7.5 \), so

\[
A = P \left(1 + \frac{r}{n}\right)^{nt} = 4500 \left(1 + \frac{0.165}{4}\right)^{4 \cdot 7.5} = \$15,130.84;
\]

for (D), \( P = 7500, r = 0.185, n = 4, \) and \( t = 9.5 \), so

\[
A = P \left(1 + \frac{r}{n}\right)^{nt} = 7500 \left(1 + \frac{0.185}{4}\right)^{4 \cdot 9.5} = \$41,803.16.
\]

Grading: +3 points for substituting values, +2 points for calculation. Grading for mistakes: −1 point for \( r = 0.0185 \) or 18.5; −2 points if the initial amount was added; −2 points for the wrong formula.
6. In 1999, the population of the town of Gilbert, AZ, was $\{105,000 \atop 101,000\}$. This year, 2001, Gilbert’s population is 117,000. Assume that the population of Gilbert grows according to the exponential model

$$P(t) = Ae^{kt},$$

where $t$ is the number of years since 1999.

a. [5 pts] Find the exact exponential model that models the population of Gilbert as a function of time $t$. (That is, find $A$ and $k$.)

Solution: Values need to be found for $A$ and $k$. These values need to satisfy $P(0) = 105000$ (for C) and $P(2) = 117000$. The first equation implies that $105000 = Ae^{k \cdot 0} = A$, and the second implies that $117000 = 105000e^{2k}$. Solving for $k$:

$$\frac{117}{105} = \frac{117000}{105000} = e^{2k}, \quad \text{or} \quad 2k = \ln\left(\frac{117}{105}\right),$$

so $k = \frac{1}{2} \ln\left(\frac{117}{105}\right) \approx 0.05410679$. Thus $P(t) = 105000e^{0.05410679t}$.

For (D), the process is the same, except that $P(0) = 101000$ and thus $k = \frac{1}{2} \ln\left(\frac{117}{101}\right) \approx 0.07352676$, so $P(t) = 101000e^{0.07352676t}$.

Grading: +2 points for $A$, +3 points for $k$. Grading for common mistakes: +3 points (total) if only the equations for $P(0)$ and $P(2)$ are given; -3 points for $P \cdot 2 = A^{2k}$ (function notation mistaken for multiplication).

b. [5 pts] Gilbert’s webpage predicts that in the year 2005, the population will be 130,000. From the model you just found, predict the population of Gilbert in the year 2005.

Solution: The year 2005 corresponds to $t = 2005 - 1999 = 6$. The answer is then $P(6)$, which is $105000e^{0.05410679 \cdot 6} \approx 145,271$ (for C) or approximately 157,005 (for D).

Grading: +2 points for $t = 6$; +3 points for evaluating $P(6)$. Grading for common mistakes: -3 points if $P(t)$ was assumed to be linear.

c. [5 pts] According to this model, when will the population reach half a million (500,000)?

Solution: The answer is the value of $t$ such that $P(t) = 500000$. For C, this involves solving the equation

$$500,000 = 105,000e^{0.05410679t}, \quad \text{then} \quad \frac{500}{105} = e^{0.05410679t},$$

$$\ln\left(\frac{500}{105}\right) = 0.5410679t,$$

so $t = \frac{1}{0.5410679} \ln\left(\frac{500}{105}\right) \approx 28.84$. This corresponds to the “date” 1999 + 28.84 = 2027.84, which occurs during the year 2027.

For (D), the process is the same, except that then $t = \frac{1}{0.0735267} \ln\left(\frac{500}{101}\right) \approx 21.75$. This occurs during the year 2020.

Grading: +2 points for the equation $P(t) = 500000$, +2 points for solving for $t$, +1 point for finding the year. Grading for common mistakes: -3 points if the model was assumed to be linear.
7. [5 pts] Evaluate the expression \[
\begin{pmatrix}
\log_4 \frac{1}{64} \\
\log_4 \frac{1}{16}
\end{pmatrix}
\]

Solution: The quick way to do this (for C) is to notice that \(64 = 4^3\), so
\[
\log_4 \frac{1}{64} = \log_4 1 - \log_4 64 = 0 - \log_4 4^3 = -3 \log_4 4 = -3 \cdot 1 = -3.
\]
The change of base formula can also be used:
\[
\log_4 \frac{1}{64} = \frac{\log_{10} \frac{1}{64}}{\log_{10} 4} \approx -1.806179974 = 3.
\]
For (D),
\[
\log_4 \frac{1}{16} = \log_4 1 - \log_4 16 = 0 - \log_4 4^2 = -2 \log_4 4 = -2.
\]
Grading for common mistakes: +2 points total if the formula was converted to exponential form.

8. [5 pts] Rewrite the expression \[
\begin{pmatrix}
5^3 = 125 \\
3^4 = 81
\end{pmatrix}
\]

in logarithmic form.

Solution: For (C): \(\log_5 125 = 3\); for (D): \(\log_3 81 = 4\).

Grading for common mistakes: -2 points if two numbers were in the wrong position; -3 points if no number was in the correct position; -1 point for \(\log_3 81 \neq 4\).

9. Consider the function given by \(y = \begin{pmatrix} e^{4x} \\ e^{2x} \end{pmatrix} \).

a. [5 pts] Graph this function and its inverse function on the same set of axes.

Solution: The inverse function is the reflection of the function \(y = e^{4x}\) about the line \(y = x\). See the webpage.

Grading: -2 points for \(\ln 2x, \log_3 x, \ln 4x, \log_4 x\); -3 points for \(e^{4x}, e^{-4x}, -e^{-4x}, -e^{4x}, e^{2x}, e^{-2x}, -e^{2x}, -e^{-2x}\).

b. [5 pts] State the inverse function.

Solution: To find the inverse of the function \(y = e^{4x}\) (version C), solve for \(x\) in terms of \(y\):
\[
\ln y = \log_e y = 4x,
\]
so \(x = \frac{1}{4} \ln y\). Thus the inverse function is \(g(x) = \frac{1}{4} \ln x\). For (D), the inverse function is \(\frac{1}{2} \ln x\).

Grading: -1 point for \(\ln 2x, \log_2 x, \ln 4x, \log_4 x\); -3 points for \(e^{4x}, e^{-4x}, -e^{-4x}, -e^{4x}, e^{2x}, e^{-2x}, -e^{2x}, -e^{-2x}\).

c. [5 pts] Label the functions, asymptotes and intercepts.

Solution: There is only one asymptote for the original function, \(y = 0\), and the original function intersects the \(y\)-axis at \((0, 1)\) (and nowhere else). The inverse function has a vertical asymptote \(x = 0\), and it intersects the \(x\)-axis at \((1, 0)\).

Grading: +1 point for labelling the functions, +1 point for each asymptote, and +1 point for each intercept.
10. Using the properties of logarithms,

a. [5 pts] Write the following as a sum, difference, or multiple of logarithms:

\[
\begin{align*}
&\ln \left( \frac{5x}{y^3} \right) \\
&\ln \left( \frac{6x}{y^4} \right)
\end{align*}
\]

Solution: For (C):
\[
\ln \left( \frac{5x}{y^3} \right) = \ln(5x) - \ln(y^3) = \ln(5) + \ln x - 3 \ln y.
\]
For (D):
\[
\ln \left( \frac{6x}{y^4} \right) = \ln(6x) - \ln(y^4) = \ln(6) + \ln x - 4 \ln y.
\]
Grading: +3 points for the first equality, +2 points for the second (one point for splitting up 5x or 6x, and one point for rewriting \( \ln y^3 \) or \( \ln y^4 \)).

b. [5 pts] Write as a single logarithm:

\[
\begin{align*}
&\frac{2}{5} \ln x + 2 \ln(x + 4) - 6 \ln y - 4 \ln z \\
&\frac{6}{7} \ln x + 9 \ln(x + 1) - 3 \ln y - 2 \ln z
\end{align*}
\]

Solution: The best way to write this as a single logarithm is to group together the terms which are being added, and group together the ones which are being subtracted. Then, for (C),
\[
\frac{2}{5} \ln x + 2 \ln(x + 4) - 6 \ln y - 4 \ln z = \ln x^{2/5} + \ln(x + 4)^2 - \ln y^6 - \ln z^4
\]
\[
= \left[ \ln x^{2/5} + \ln(x + 4)^2 \right] - \left[ \ln y^6 + \ln z^4 \right]
\]
\[
= \ln x^{2/5}(x + 4)^2 - \ln y^6 z^4 = \ln \left( \frac{x^{2/5}(x + 4)^2}{y^6 z^4} \right);
\]
for (D):
\[
\frac{6}{7} \ln x + 9 \ln(x + 1) - 3 \ln y - 2 \ln z = \ln x^{6/7} + \ln(x + 1)^9 - \ln y^3 - \ln z^2
\]
\[
= \left[ \ln x^{6/7} + \ln(x + 1)^9 \right] - \left[ \ln y^3 + \ln z^2 \right]
\]
\[
= \ln x^{6/7}(x + 1)^9 - \ln y^3 z^2 = \ln \left( \frac{x^{6/7}(x + 1)^9}{y^3 z^2} \right).
\]
Grading for common mistakes: +3 points total for \( \frac{\ln x^{2/5} \ln(x + 4)^2}{\ln y^6 \ln z^4} \) (C) or \( \frac{\ln x^{6/7} \ln(x + 1)^9}{\ln y^3 \ln z^2} \) (D); −1 point if the ln was omitted;.
11. In the Fall Semester, students in a psychology class with a size of 100 were told they would be given a memory test the following semester. The test was graded out of 100 points. The test went like this: In the following Spring Semester, the students were shown pictures of their classmates from the previous Fall. They were to identify the picture with the name of the student. The students were given the same test for the next 5 months. The average score of the memory test was given by the model:

\[ f(t) = \begin{cases} 
80 - 5 \log_{10}(t + 3) & , \quad 0 \leq t \leq 5, \\
78.2 - 6 \log_{10}(t + 7) & 
\end{cases} \]

where \( t \) is the time in months.

a. [5 pts] What was the average score of the original exam?

Solution: The average score of the original exam is \( f(0) \). For (C), this is \( 80 - 5 \log_{10}(0 + 3) \approx 77.6 \); for (D), it is \( 78.2 - 6 \log_{10}(0 + 7) \approx 73.1 \).

Grading: +2 points for \( t = 0 \); +3 points for evaluating \( f(0) \). Grading for common mistakes: –1 point for arithmetic; –2 points for a problem with evaluating the \( \log_{10} \) factor; –2 points for \( 80 \) (C) or \( 78.2 \) (D).

b. [5 pts] What would you predict, from this model, the average score of the exam to be after two years?

Solution: Since \( t \) is the time in months since the initial exam, \( t = 2 \times 12 = 24 \). Then the (expected) average score after two years is \( f(24) \), which is \( 80 - 5 \log_{10}(27) \approx 72.8 \) (for C) or \( 78.2 - 6 \log_{10}(31) \approx 69.25 \) (for D).

Grading: +2 points for \( t = 24 \), +3 points for evaluating \( f(t) \). Grading for common mistakes: –1 point for \( t = 2 \).

c. [5 pts] When would you expect the average score to be \( \{ 65 \ \ | \ \ 60 \} \%? \)

Solution: This problem had some unexpected leeway. The intended objective was to determine the value of \( t \) for which \( f(t) = 65 \) (for C) or 60 (for D). For version C, this means solving the equation \( 80 - 5 \log_{10}(t + 3) = 65 \), which is done as follows:

\[
\begin{align*}
\text{Version C} & \quad \text{Version D} \\
80 - 5 \log_{10}(t + 3) &= 65 & 78.2 - 6 \log_{10}(t + 7) &= 60 \\
15 &= 5 \log_{10}(t + 3) & 18.2 &= 6 \log_{10}(t + 7) \\
3 &= \log_{10}(t + 3) & 273/90 &= \log_{10}(t + 7) \\
t + 3 &= 10^3 & t + 7 &= 10^{273/90} \\
t &= 10^3 - 3 = 997 & t &= 10^{273/90} - 7 \approx 1072.8
\end{align*}
\]

(Note that 997 months is approximately 83 years, and 1073 months is approximately 89\(\frac{1}{2}\) years. This shows that the model has probably been stretched too far; how many 101-year-olds can remember 65% of their classmates at age 18?)

Some people evidently evaluated \( f(t) \) for increasing values of \( t \) (or graphed the function and traced or zoomed in on the graph) until a value of 65.9 was obtained, and the value of \( t \) was then noted. For version C of the test, this occurs at \( t = 663 \), and for version D at \( t = 729 \). Full credit
was given for any value of \( t \) in the appropriate range, provided there was enough supporting work.

Grading: +2 points for the equation \( f(t) = 65 \) (or 60), +3 points for finding \( t \). Grading for common errors: −2 points if \( f(t) \) was assumed to be linear; +3 points total for a correct answer but no work. Full credit was given for any value of \( t \) between 663 and 997 for version C, and for any value of \( t \) between 729 and 1073 for version D.

12. [5 pts] Complete the table below for a savings account in which interest is compounded continuously.

<table>
<thead>
<tr>
<th>Initial Investment</th>
<th>Annual % Rate</th>
<th>Time to Double</th>
<th>Amount after ( { 10 } ) years</th>
</tr>
</thead>
<tbody>
<tr>
<td>${ 4,000 } { 6,000 }</td>
<td>{} 3 } { 5 } years</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Solution: Since the interest is compounded continually, we need to use the formula \( A = Pe^{rt} \). To find the amount after 10 years (8 years for version D), we will need the annual interest rate, so we have to find the “Annual % Rate” first. The doubling time of our investment is 3 years (5 years for D), so we need to solve the equation

\[
8000 = 4000 \cdot e^{3r}
\]

for C, and

\[
12000 = 6000 \cdot e^{5r}
\]

for D. This we do as follows:

<table>
<thead>
<tr>
<th>Version C</th>
<th>Version D</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 8000 = 4000 \cdot e^{3r} )</td>
<td>( 12000 = 6000 \cdot e^{5r} )</td>
</tr>
<tr>
<td>( 2 = e^{3r} )</td>
<td>( 2 = e^{5r} )</td>
</tr>
<tr>
<td>( 3r = \ln 2 )</td>
<td>( 5r = \ln 2 )</td>
</tr>
<tr>
<td>( r = \frac{1}{3} \ln 2 \approx 0.2310 )</td>
<td>( r = \frac{1}{5} \ln 2 \approx 0.13863 )</td>
</tr>
</tbody>
</table>

Thus 23.1% goes under “Annual % Rate” for version C, and 13.9% goes under “Annual % Rate” for version D.

To find the amount after a certain number of years, we just need to find \( A \):

\[
A = 4000 \cdot e^{0.02310 \cdot 10} = \$40,297.70
\]

for (C), and

\[
A = 6000 \cdot e^{0.13863 \cdot 8} = \$18,188.60
\]

for (D). This number goes under “Amount after \( X \) years.”

Grading: +3 points for the annual interest rate, +2 points for the amount after 8 or 10 years. Grading for common mistakes: −2 points if the amount after 8 (or 10) years was not in dollars (but some other units). Round-off errors resulted in slightly different answers; these were checked as they occurred.

MAT 117 Website: \texttt{http://math.la.asu.edu/~checkman/117/}