MATH 243E Test #3 Solutions

(1) Find a recurrence relation for the number of bit strings of length $n$ that contain a pair of consecutive 0s. You do not need to solve this recurrence relation. (Hint: Consider the following cases: the last bit is a 1, the last two bits are 10, or the last two bits are 00. Don’t forget the initial conditions.) (25 points)

Solution: Let $a_n$ be the number of strings of length $n$ containing a pair of consecutive 0s. Using the hint, we know that every string of length $n$ either has its last bit being a 1, its last two bits being 10, or its last two digits being 00. We consider each case in turn.

If the last bit is a 1, then the string with this 1 removed must have a consecutive pair of 0s. Hence exactly $a_{n-1}$ of the $a_n$ strings we are considering end in a 1. Similarly, if the last two bits are 10, then the string with these bits removed still contains a consecutive pair of 0s. Hence exactly $a_{n-2}$ of the $a_n$ strings we are considering end in 10.

Something different happens if the last two bits of the string we are considering are 00; the smaller string can be anything. Hence, exactly $2^{n-2}$ of the $a_n$ strings we are considering end in 00.

Because these cases are mutually exclusive, we have

$$a_n = a_{n-1} + a_{n-2} + 2^{n-2}, \text{ for } n \geq 2$$

(or for $n \geq 3$ if you feel squeamish about dealing with strings of length zero).

We still need to supply initial conditions; counting shows that $a_1 = 0$ and $a_2 = 1$. Hence the recurrence relation for $a_n$ is

$$a_n = a_{n-1} + a_{n-2} + 2^{n-2}, \text{ for } n \geq 2,$$

$$a_1 = 0,$$

$$a_2 = 1.$$

Grading: −5 points for using $a_{n-2}$ instead of $2^{n-2};$ −5 points for forgetting the initial conditions.
(2) Solve the following recurrence relation. (25 points)

\[ d_n - 6d_{n-1} - 16d_{n-2} = 2^n, \quad n \geq 2, \]
\[ d_0 = 1, \]
\[ d_1 = 5. \]

Solution: This is a linear, nonhomogeneous recurrence relation, so we have three steps to perform:

(1) Solve the homogeneous version: \( h_n - 6h_{n-1} - 16h_{n-2} = 0. \)

The auxiliary equation for this (linear, homogeneous) recurrence is \( r^2 - 6r - 16 = 0 \) or \( (r - 8)(r + 2) = 0. \) Hence \( r = 8 \) or \(-2,\) and the solution to the homogeneous equation is

\[ h_n = c_1 \cdot (-2)^n + c_2 \cdot 8^n, \]
for some constants \( c_1 \) and \( c_2. \)

(2) Find one solution \( p_n \) satisfying the original recurrence.

The right-hand side of the original recurrence is \( 2^n, \) a constant polynomial times \( 2^n. \) Because 2 is not a root of the auxiliary equation of the homogeneous recurrence, our deep theorem tells us that there is a solution of the form \( p_n = C \cdot 2^n, \) where \( C \) depends on the recurrence. Because \( p_n \) must satisfy the recurrence, we must have

\[ C \cdot 2^n - 6C \cdot 2^{n-1} - 16C \cdot 2^{n-2} = 2^n. \]

Dividing both sides by \( 2^n, \) we have \( C - \frac{6}{2} C - \frac{16}{4} C = 1, \) or \( C = -\frac{1}{6}. \)

Hence our particular solution is

\[ p_n = -\frac{1}{6} \cdot 2^n. \]

(3) Add them together and plug in the initial conditions.

Hence \( d_n = c_1 \cdot (-2)^n + c_2 \cdot 8^n - \frac{1}{6} \cdot 2^n. \) Plugging in the initial conditions, we have \( 1 = d_0 = c_1 + c_2 - \frac{1}{6} \) (and hence \( c_1 + c_2 = \frac{7}{6} \)) and \( 5 = d_1 = -2c_1 + 8c_2 - \frac{1}{3}, \) or \(-2c_1 + 8c_2 = \frac{16}{3}. \) Solving these equation yields \( c_1 = \frac{2}{5} \) and \( c_2 = \frac{23}{30} \) (more work than I had anticipated). Thus

\[ d_n = \frac{2}{5} \cdot (-2)^n + \frac{23}{30} \cdot 8^n - \frac{1}{6} \cdot 2^n. \]

Grading: -10 points for no particular solution; -5 points for not finding \( c_1 \) and \( c_2. \)
(3) Recall that an \textit{onto} function is a function \( f : A \to B \) such that for every element \( b \in B \), there exists an element \( a \in A \) with \( f(a) = b \). In this problem we will count the number of onto functions from \( A \) to \( B \), where \( A = \{a, b, c, d, e\} \) and \( B = \{1, 2, 3\} \).

(a) How many functions \( g : S \to T \) are there, if \( S \) has \( m \) elements and \( T \) has \( n \) elements? (5 points)

\textit{Answer:} \( n^m \). Grading: +3 points for anything else, as long as it was used for \( (b) \).

Let the set \( A_i \) consist of all functions \( f : A \to B \) such that \( i \) is not in the range of \( f \), where \( i \) is 1, 2, or 3. (For instance, \( A_1 \) is the set of all functions \( f \) from \( A \) to \( B \) which do \textbf{not} have 1 in the range of \( f \); i.e., \( f \) is then a function from \( A \) to \( B \setminus \{1\} = \{2, 3\} \).)

(b) What is \( |A_1 \cup A_2 \cup A_3| \)? (15 points)

\textit{Solution:} Inclusion-exclusion tells us that

\[ |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|. \]

Since \( A_1 \) is the set of all functions with domain \( A \) and range \( \{2, 3\} \), \( |A_1| \) is just the number of such functions, or \( 2^5 = 32 \). Similarly, \( |A_2| = |A_3| = 32 \) as well.

Now, what is \( A_1 \cap A_2 \)? It is a subset of the set of functions with domain \( A \), but what is the range of a function \( f \in A_1 \cap A_2 \)? Neither 1 nor 2 can be in the range of \( f \), so the range of \( f \) is only \( \{3\} \). Hence \( A_1 \cap A_2 \) is the set of all functions with domain \( A \) and range \( \{3\} \), and thus \( |A_1 \cap A_2| = 1^5 = 1 \). Similarly, \( |A_1 \cap A_3| = |A_2 \cap A_3| = 1 \) as well.

Now what about \( A_1 \cap A_2 \cap A_3 \)? Thinking along the lines above, it contains precisely the functions from \( A \) to \( B \), whose range does not include 1, 2, or 3. There are no such functions, since \( B = \{1, 2, 3\} \), so \( A_1 \cap A_2 \cap A_3 = \emptyset \), and it has size 0.

Plugging these values into our inclusion-exclusion equation, we have

\[ |A_1 \cup A_2 \cup A_3| = 32 + 32 + 32 - 1 - 1 - 1 + 0 = 93. \]

Grading: full credit if the number of functions computed was consistent with the answer to (a).

(c) What does the set \( A_1 \cup A_2 \cup A_3 \) have to do with onto functions from \( A \) to \( B \)?

How many onto functions are there from \( A \) to \( B \)? (5 points)

\textit{Solution:} The set \( A_1 \cup A_2 \cup A_3 \) consists of all functions from \( A \) to \( B \) which are \textbf{not} onto, since at least one of 1, 2, or 3 is missing from the range of a function in this union. Hence the number of onto functions from \( A \) to \( B \) is the total number of functions from \( A \) to \( B \) minus the size of the set \( A_1 \cup A_2 \cup A_3 \) (the “bad functions”), or \( 3^5 - 93 = 160 \).

Grading: +3 points for saying that \( A_1 \cup A_2 \cup A_3 \) is the set of all onto functions.
(4) How many integers between 0 and 9,999,999 (inclusive) are there whose digits add up to 23? (25 points)

Solution: The question can be viewed as the following problem: How many solutions are there to the equation

\[
x_1 + x_2 + \cdots + x_7 = 23, \text{ where } 0 \leq x_i \leq 9, \text{ for all } i.
\]

Without the upper bounds, the answer would be \( \binom{23 + 7 - 1}{23} \). This answer includes solutions where at least one of the \( x_i \)'s is at least 10. Let us call such a solution “bad”; the question is now: How many bad solutions are there?

Let \( A_i \) consist of all solutions to (*) with \( x_i \geq 10 \). Then the number of bad solutions to (*) is \( |A_1 \cup A_2 \cup \cdots \cup A_7| \), which is equal to

\[
\sum_{1 \leq i \leq 7} |A_i| - \sum_{1 \leq i < j \leq 7} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 7} |A_i \cap A_j \cap A_k| - \cdots,
\]

where the last term is \( A_1 \cap A_2 \cap \cdots \cap A_7 \). Since all the bounds on the \( x_i \)'s are the same, the above expression simplifies to

\[
\binom{7}{1} |A_1| - \binom{7}{2} |A_1 \cap A_2| + \binom{7}{3} |A_1 \cap A_2 \cap A_3| - \cdots.
\]

So how big is \( A_1 \)? \( A_1 \) consists of all solutions of (*) with \( x_1 \geq 10 \). If we let \( x'_1 = x_1 - 10 \), we have \( x'_1 + x_2 + x_3 + \cdots + x_7 = 23 - 10 = 13 \), where \( x'_1, x_2, x_3, \ldots, x_7 \geq 0 \). The number of solutions to this equation is equal to the size of \( A_1 \); hence \( |A_1| = \binom{13 + 7 - 1}{13} = \binom{19}{13} \).

Similarly, for a solution to belong to \( A_1 \cap A_2 \), it must have \( x_1, x_2 \geq 10 \). If we let \( x'_1 = x_1 - 10 \) and \( x'_2 = x_2 - 10 \), we see that \( A_1 \cap A_2 \) has \( \binom{3 + 7 - 1}{3} = \binom{9}{3} \) elements.

Clearly, \( A_1 \cap A_2 \cap A_3 \) must be empty, because otherwise the first three digits would add up to at least 30. Hence the number of bad solutions is

\[
\binom{7}{1} \cdot \frac{19}{13} - \binom{7}{2} \cdot \frac{9}{3},
\]

and the number of good solutions to (*) is

\[
\binom{23}{23} - \binom{7}{1} \cdot \frac{19}{13} + \binom{7}{2} \cdot \frac{9}{3} = 286.860.
\]

Grading: +5 points for just setting up \( A_i \); +5 points for using 6 instead of 7, the rest being okay.
(5) Suppose that $C_n$ satisfies the homogeneous linear recurrence relation

$$C_{n+5} = 5C_{n+4} - 7C_{n+3} - C_{n+2} + 8C_{n+1} - 4C_n, \quad \text{for } n \geq 5.$$ 

What is the general form for the solution to this recurrence? (Hint: The auxiliary equation for this recurrence is $(r + 1)(r - 1)^2(r - 2)^2 = 0$.) (25 points)

Solution: The roots of the auxiliary equation are $-1$ (with multiplicity 1), $1$ (with multiplicity 2), and 2 (with multiplicity 2), so the general form of the solution to the recurrence is

$$C_n = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot 1^n + \alpha_3 \cdot n \cdot 1^n + \alpha_4 \cdot 2^n + \alpha_5 \cdot n \cdot 2^n,$$

where $\alpha_1, \ldots, \alpha_5$ are constants (yet to be determined).

Grading: +15 points if the roots were interpreted to be $-1, \pm 1, \pm \sqrt{2}$ (and the rest of the problem done correctly); +20 points if $\alpha_2 \cdot n + \alpha_3 \cdot n^2$ showed up instead of $\alpha_2 + \alpha_3 \cdot n$; +20 points if the $\alpha_2 \cdot 1^n$ term and/or the $\alpha_4 \cdot 2^n$ term was omitted.