(1) How many palindromes (integers whose decimal digit strings are the same read from left to right as from right to left) are there between 100 and 10,000? (15 points)

Solution: There are two types of integers between 100 and 10,000 which could be palindromes: 3-digit and 4-digit numbers. For 3-digit numbers, the first (hundreds') and third (ones') digits must be the same, and non-zero; there are 9 possible choices for the first (and only 1 for the third). For the second (tens') digit, any digit 0–9 is acceptable, and there are 10 choices for that one. Thus the total number of 3-digit palindromes is \(9 \cdot 10 \cdot 1 = 90\).

Similarly, the number of 4-digit palindromes is \(9 \cdot 10 \cdot 1 \cdot 1 = 90\), since the first (thousands') and fourth (ones') digits have to be the same and nonzero, and the second (hundreds') and third (tens') digits have to be the same as well, and 0–9 are the acceptable digits for these positions. Then the total number of palindromes between 100 and 10,000 is \(90 + 90 = 180\).

Grading: +5 points for the 3-digit case, +5 points for the 4-digit case, and +5 points. Common mistakes: +4 points for \(9 \cdot 9 \cdot 1\) in either case; +2 points for \(9 + 9 + 1\) in either case. +3 points for work that showed promise.

(2) Use induction to prove that \(1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}\), for all positive integers \(n\). (15 points)

Solution: Let \(P(n)\) denote the statement \(1 \cdot 2 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}\). (Note that \(P(n)\) is not just \(1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1)\); \(P(n)\) must be a “sentence.”)

We need to show that (1): \(P(1)\) is true, and (2): \(P(k) \Rightarrow P(k + 1)\), for \(k \geq 2\).

Part (1) is easy enough: \(1 \cdot 2 = 2 = \frac{1 \cdot (1 + 1) \cdot (1 + 2)}{3}\). Now we show that \(P(k + 1)\) is true, if \(P(k)\) is, by evaluating the left-hand side of \(P(k + 1)\):

\[
1 \cdot 2 + \cdots + (k + 1)(k + 2) = 1 \cdot 2 + \cdots + k(k + 1) + (k + 1)(k + 2)
= \frac{k(k + 1)(k + 2)}{3} + (k + 1)(k + 2), \text{ because } P(k) \text{ is true;}
= (k + 1)(k + 2) \left( \frac{k}{3} + 1 \right) = \frac{(k + 1)(k + 2)(k + 3)}{3}.
\]

Grading: +5 points for showing that \(P(1)\) is true; +5 points for writing down \(P(k + 1)\); +5 points for proving \(P(k + 1)\).
(3) Suppose that a nonnegative integer $N$ between 0 and 999,999 (inclusive) is chosen at random. What is the probability that the sum of the digits of $N$ is equal to 25? (Hint: Think of these numbers as being strings of digits of length six.) (20 points)

**Solution:** The probability that the sum of the digits of $N$ adds up to 25 is equal to the number of integers (between 0 and 999,999) whose digits add up to 25 divided by the number of integers between 0 and 999,999. The latter is 1,000,000.

To find the number of integers whose digits add up to 25, let $x_1, x_2, \ldots, x_6$ be the digits of $N$ (left to right). Note that we can have $x_1 = 0$, because we are thinking of the numbers as strings of digits. We then want the number of solutions to the equation

$$x_1 + x_2 + \cdots + x_6 = 25,$$

where $0 \leq x_i \leq 9$, for all $i$.

Without the upper bound, the answer would be \(\binom{25 + 6 - 1}{25}\), because the lower bound is already 0. To find the number of solutions to the equation above, we will count the number of “bad” solutions and subtract this from the total number of solutions. Define the set $A_i$ to be the set of all solutions to (*) such that $x_i \geq 10$: then the number of bad solutions is $|A_1 \cup A_2 \cup \cdots \cup A_6|$.

The inclusion-exclusion formula implies that

$$|A_1 \cup A_2 \cup \cdots \cup A_6| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots.$$

Note that since the bounds on $x_i$ are the same for all $i$, this sum evaluates to

$$|A_1 \cup A_2 \cup \cdots \cup A_6| = \binom{6}{1} \cdot |A_1| - \binom{6}{2} \cdot |A_1 \cap A_2| + \binom{6}{3} \cdot |A_1 \cap A_2 \cap A_3| - \cdots.$$

Now we find $A_1$: These are all solutions to (*) with $x_1 \geq 10$. If we let $x_1' = x_1 - 10$, we see that $x_1' + x_2 + \cdots + x_6 = 25 - 10 = 15$ and $x_1', x_2, \ldots, x_6 \geq 0$. The number of solutions to this equation is \(\binom{15 + 6 - 1}{15}\), and so $|A_1| = \binom{20}{5}$.

Similarly, $A_1 \cap A_2$ consists of all solutions to (*) such that $x_1, x_2 \geq 10$. If we let $x_1'' = x_1 - 10$ and $x_2'' = x_2 - 10$, we see that we are counting solutions to the equation $x_1'' + x_2'' + x_3 + \cdots + x_6 = 25$, where $0 \leq x_i'' \leq 9$, for all $i$, which is just \(\binom{5 + 6 - 1}{5}\).

Note that $A_1 \cap A_2 \cap A_3 = \emptyset$ so the inclusion-exclusion formula in this case gives the number of bad solutions as \(\binom{30}{5} - \binom{6}{1} \cdot \binom{20}{5} + \binom{6}{2} \cdot \binom{10}{5}\) = 53,262,

and the probability that the sum of the digits in $N$ is thus \(\frac{53,262}{1,000,000}\).

Grading: +5 points for the definition of probability; +5 points for the number \(\binom{30}{5}\).
(4) Let $S$ be a set of five (distinct) positive integers which are all divisors of 24. Prove that there are two numbers in $S$ whose product is 24. Is the previous statement still true if “five” is replaced by “four”? (15 points)

Solution: The positive integers which are divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. I will set up four bins (pigeonholes):

\[
\begin{array}{|c|c|c|c|}
\hline
1, 24 & 2, 12 & 3, 8 & 4, 6 \\
\hline
\end{array}
\]

(That is, only 1 and 24 can be put in the first bin, 2 and 12 in the second, etc.)

If five distinct divisors of 24 are put into these bins, one bin has at least \(\lceil \frac{5}{4} \rceil = 2\) numbers in it. But these two numbers multiply together to yield 24.

The statement is not true if \(|S| = 4\); the set \(\{1, 2, 3, 4\}\) is a counterexample.

(5) Prove that if $x$ and $y$ are real numbers, then \(\max \{x, y\} \cdot \min \{x, y\} = xy\). (15 points)

Solution: If $x \geq y$, then \(\max \{x, y\} = x\) and \(\min \{x, y\} = y\); but then

\[\max \{x, y\} \cdot \min \{x, y\} = xy.\]

If $y \geq x$, then \(\max \{x, y\} = y\) and \(\min \{x, y\} = x\); but then

\[\max \{x, y\} \cdot \min \{x, y\} = yx = xy\]

as well. Since for any two real numbers $x$ and $y$, $x \geq y$ or $y \geq x$, these cases exhaust all possibilities for $x$ and $y$. 

3
(6) Describe an algorithm to search for the number \( x \) in the array \( a \). Specifically, describe a recursive function \( \text{BSEARCH}(a, x, m, n) \) which returns TRUE if \( x \) is equal to one of \( a[m], a[m+1], \ldots, a[n] \), and returns FALSE otherwise. Make sure your algorithm works when \( n = m + 1 \). How would you use \( \text{BSEARCH} \) as a subroutine to design an algorithm that determines whether the number \( x \) is in the array \( a \), if \( a \) has \( n \) elements? (You may assume that the array \( a \) is sorted.) (20 points)

Solution: The following routine (written in pseudocode) will work:

```pseudocode
function BSEARCH (a, x, m, n)
    if (m = n)
        if (a[m] = x) return TRUE
        else return FALSE
    else
        begin
            mid := (m + n) / 2
            if (a[mid] < x) return BSEARCH(a, x, mid + 1, n)
            else if (a[mid] > x) return BSEARCH (a, x, m, mid - 1)
            else return TRUE
        end
    end
```

The function call \( \text{BSEARCH}(a, x, 1, n) \) will return TRUE if the number \( x \) is in the array \( a \) and FALSE otherwise.

Grading: +5 points for a search algorithm; +10 for a non-recursive binary search or a recursive “non-binary” search; +15 points if the algorithm “almost” works.