MATH 119-P Test #3 Solutions

(1) Suppose that a die with faces 1, 2, ..., 6 is weighted so that the probabilities that
$P(1) = P(4) = P(6), P(2) = P(3) = P(5),$ and $P(2) = 2 \cdot P(1),$ where $P(i)$ is the probability that face $i$ comes up on top.

(a) Find $P(1), P(2), ..., P(6).$ (10 points)
(b) Find $P(\{1, 2, 3\}).$ (5 points)

Solution: Note that $P(i)$ is a probability distribution, so


If we let $x = P(1), P(4) = x, P(6) = x, P(2) = 2x, P(3) = P(2) = 2x,$ and $P(5) = P(2) = 2x.$ If we make these substitutions, we get

$$x + 2x + 2x + 2x + x = 1$$
$$9x = 1,$$

or $x = \frac{1}{9}.$ Then we have

$$P(1) = P(4) = P(6) = \frac{1}{9},$$

and

$$P(2) = P(3) = P(5) = \frac{2}{9},$$

the answer to part (a). To solve (b), note that

$$P(\{1, 2, 3\}) = P(1) + P(2) + P(3) = \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{5}{9}.$$

Grading: For (a), +3 points for $P(1) = \frac{1}{6}$ and so on, and +6 points for any attempt similar to the one above; for (b), +3 points if the answers to (a) were multiplied.

(2) Let $U$ denote the set of all Senators, $D$ the set of all Senators who are Democrats, $W$ the set of all Senators who are women, and $I$ the set of all incumbants (Senators who were also Senators before the election).

(a) Describe the set $D \cap \overline{W}$ in words. (5 points)

Answer: The set of all male Democrats.

(b) How would you represent the set of all women who have become Senators for the first time? (10 points)

Answer: $W \cap \mathcal{T}.$ Scoring: +3 points for $D \cap W; +7$ points for $W \cup \mathcal{T}$ or $W \cap I.$
(3) A survey is taken of 1000 people, where every person is asked their religion and voting preference. The results of this poll are shown in the table below.

<table>
<thead>
<tr>
<th></th>
<th>Democrat (D)</th>
<th>Republican (R)</th>
<th>Independent (I)</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Catholic (C)</td>
<td>200</td>
<td>155</td>
<td>45</td>
<td>400</td>
</tr>
<tr>
<td>Protestant (P)</td>
<td>198</td>
<td>77</td>
<td>35</td>
<td>310</td>
</tr>
<tr>
<td>Jewish (J)</td>
<td>102</td>
<td>83</td>
<td>105</td>
<td>290</td>
</tr>
<tr>
<td>Totals</td>
<td>500</td>
<td>315</td>
<td>185</td>
<td>1000</td>
</tr>
</tbody>
</table>

(a) Suppose someone from the survey is picked at random. What is the probability that they are Jewish, given that they are a Democrat? What is the probability that they are a Democrat, given that they are Jewish? (2 × 5 points)

Solution:

\[ P(J|D) = \frac{P(J \cap D)}{P(D)} = \frac{\frac{102}{1000}}{\frac{500}{1000}} = \frac{102}{500}, \text{ and} \]

\[ P(D|J) = \frac{P(D \cap J)}{P(J)} = \frac{\frac{102}{290}}{\frac{1000}{1000}} = \frac{102}{290}. \]

(b) Are the events D and R independent? Are the events D and C independent? (2 × 5 points)

Solution: For (b), we need to check whether the equality \( P(E \cap F) = P(E) \cdot P(F) \) holds in each case. The first question asks whether

\[ P(D \cap R) = P(D) \cdot P(R) \text{ or} \]

\[ 0 = \frac{500}{1000} \cdot \frac{315}{1000}, \]

since no one can be a Democrat and a Republican at the same time. This equality is not true, so \( D \) and \( R \) are not independent. The second question asks whether

\[ P(D \cap C) = P(D) \cdot P(R) \text{ or} \]

\[ \frac{200}{1000} = \frac{500}{1000} \cdot \frac{400}{1000}, \]

which is true. Thus the events \( D \) and \( C \) are independent.

Grading (for (b)): +5 points for the correct answers, +3 points for one correct answer, and +0 points for no correct answers; the remaining +5 points was awarded if work was included, with +3 points if \( c(D) \) was used instead of \( P(D) \).
(4) In the PowerBall lottery, five distinct numbers between 01 and 49 are chosen, as well as the PowerBall, a number between 01 and 42. The order in which the first five numbers appear is irrelevant, and the PowerBall number need not be distinct from the others. How many PowerBall lottery tickets are possible? Express you answer as a number. (15 points)

Solution: The first part of a PowerBall lottery ticket consists of five numbers chosen out of 49, where repetition is not allowed, and order doesn't matter, so there are \( C(49, 5) \) ways to choose these numbers. There are 42 ways to choose the PowerBall number. Since any combination of the first five numbers and the PowerBall number is allowed (“the PowerBall need not be distinct from the others”), the number of possible tickets is

\[
P(49, 5) \cdot 42 = \frac{49!}{(49 - 5)!5!} \cdot 42 = 80,089,128.
\]

Grading: points were awarded for the following answers:

<table>
<thead>
<tr>
<th>Formula</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 42 \cdot P(49, 5) )</td>
<td>+10 points</td>
</tr>
<tr>
<td>( C(50, 5) \cdot C(43, 1) )</td>
<td>+12 points</td>
</tr>
<tr>
<td>( 42 \cdot C(49, 5) ) but evaluated as ( 42 \cdot P(49, 5) )</td>
<td>+12 points</td>
</tr>
<tr>
<td>( P(49, 6) )</td>
<td>+7 points</td>
</tr>
</tbody>
</table>

Furthermore, 5 points were taken away if the final answer was not a number.

(5) If three fair dice are tossed, what is the probability that the product of the numbers appearing on the top is 6? (15 points)

Solution: Because each die can come up a 1, 2, 3, 4, 5, or 6, there are six outcomes for each die. The total number of outcomes is thus \( 6 \cdot 6 \cdot 6 = 216 \). Now we need to determine which outcomes have the property that the product of the numbers on top is six. There are only nine: 123, 132, 213, 231, 312, 321, 116, 161, and 611. The probability that one of these outcomes occurs is thus \( \frac{9}{216} = \frac{1}{24} \), since each of the numbers 1, 2, \ldots, 6 comes up with probability \( \frac{1}{6} \).

Grading: +5 points saying that the probability is the ratio of the two quantities; +10 points if the probability that the sum is six was calculated; +12 points if the outcomes 116, 161, 611 were counted twice (which gives an answer of \( \frac{12}{216} \)); +10 points if the outcomes 116, 161, and 611 were omitted (giving an answer of \( \frac{6}{216} \)); +10 points for an answer of 9.
(6) Of the cars sold during the month of July, 141 had air conditioning, 141 had automatic transmissions, and 94 had power steering. 47 cars had all three of these extras. Eight cars had none of these extras. Twenty-three cars had only air conditioning; 31 cars had only automatic transmission; and 15 cars had only power steering. Fifty-nine cars had both automatic transmission and power steering.

(a) How many of these cars had both air conditioning and automatic transmission?
(b) How many cars were sold in July?

Solution: The easiest way to solve this problem is to set up a Venn diagram, as the one below:

```
\begin{center}
\begin{tikzpicture}
  \node [shape=circle,draw,minimum width=1cm] (A) at (0,0) {AC};
  \node [shape=circle,draw,minimum width=1cm] (B) at (3,0) {AT};
  \node [shape=circle,draw,minimum width=1cm] (C) at (1.5,-2) {PS};
  \node (x) at (1.3,0.5) {x};
  \node (y) at (1.3,-1.5) {y};
  \node (z) at (2.5,-1.5) {z};
  \node (23) at (0,-1) {23};
  \node (31) at (3,-1) {31};
  \node (47) at (1.5,-2) {47};
  \node (15) at (1.5,-3.5) {15};
  \node (8) at (2.5,-3.5) {8};
  \draw (A) -- (B) -- (C) -- (A);
\end{tikzpicture}
\end{center}
```

The numbers represent areas that we know about from the information, and variables represent quantities which we do not yet know. The number of cars with automatic transmission and power steering is 59; our diagram then tells us that \(47 + z\) cars with automatic transmission and power steering were sold. Hence \(47 + z = 59\), or \(z = 12\).

Now we can find \(x\) and \(y\). The total number of cars sold with automatic transmission is 141; our diagram says that the total number of cars sold with automatic transmission is \(x + 31 + 47 + z = x + 90\). Hence \(x + 90 = 141\), or \(x = 51\). The total number of cars sold with power steering is 94 and also \(15 + y + 47 + z = y + 74\); hence \(y = 94 - 74 = 20\), and we know the number of cars which are in any area of the Venn diagram above.

The answer to (a) is the sum of the regions corresponding to the set \(AC \cap AT\), or \(47 + x = 98\) cars. The answer to (b) is the sum of all the numbers in the diagram, which turns out to be 207.

Grading: +10 points for the Venn diagram setup; +5 points appiece for answering (a) and (b).
(7) (Extra Credit) Suppose that \( A \) and \( B \) are sets such that \( c(A) = 25 \), \( c(B) = 20 \), and \( c(A \cup B) = 40 \). What is \( c(A \cap \overline{B}) \)? (10 points)

Solution: A Venn diagram can be used, but \( c(A \cap B) \) needs to be calculated. the inclusion-exclusion formula gives us this:

\[
c(A \cup B) = c(A) + c(B) - c(A \cap B)
\]

\[
40 = 25 + 20 - c(A \cap B)
\]

\[
-5 = -c(A \cap B),
\]

or \( c(A \cap B) = 5 \). Now we can fill in the Venn diagram for the sets \( A \) and \( B \):

![Venn Diagram](image)

The region for the set \( A \cap \overline{B} \) has been shaded. Since it has exactly 20 elements, \( c(A \cap \overline{B}) = 20 \).

Grading: +5 points for \( c(A \cap B) = 5 \).