Let \( a_1, \ldots, a_n \) be a sequence of nonzero real numbers, exactly \( p \) of which are positive. Characterize the pairs \((n, p)\) such that exactly half of the possible products \( a_ia_ja_k \) with \( i < j < k \) are positive.

**Solution by Christopher Carl Heckman, Georgia Institute of Technology**

This isn’t too difficult. Define the numbers \( x_i \) by

\[
x_i = \begin{cases} +1, & \text{if } a_i > 0; \\ -1, & \text{if } a_i < 0. 
\end{cases}
\]

Note that \( x_ix_jx_k = +1 \) when \( a_ia_ja_k > 0 \), and \( x_ix_jx_k = -1 \) when \( a_ia_ja_k < 0 \). Notice that \( \sum_i x_i^2 = \sum_i \left( x_i^2 \right) = 3 \) for each of these is a valid solution to the original problem, for any positive \( k \).

We now solve this equality for integral values of \( n \) and \( p \). If \( n = 2p \), then exactly half of the products \( x_ix_jx_k \) are positive, and so exactly half of the products \( a_ia_ja_k \) are positive (for \( i < j < k \)). Otherwise, \( (2p - n)^2 = 3n - 2 \), or \( n^2 - (4p + 3)n + (4p^2 + 2) = 0 \). Applying the quadratic formula, we obtain

\[
n = \frac{4p + 3 \pm \sqrt{(4p + 3)^2 - 4(4p^2 + 2)}}{2} = 2p + \frac{3}{2} \pm \frac{1}{2} \sqrt{24p + 1}.
\]

For \( n \) to be rational, we need \( 24p + 1 \) to be a perfect square; for \( n \) to be an integer, we also require that \( \sqrt{24p + 1} \) to be odd, and hence \( 24p + 1 \) is an odd square. Thus we can write \( 24p + 1 = (2k + 1)^2 \), for some integer \( k \geq 0 \). Solving for \( p \), we get \( p = \frac{1}{6} k(k + 1) \), which is an integer iff \( k \equiv 0 \) or \( 2 \pmod{3} \). Substituting for \( p \), we get

\[
n = \frac{k(k + 1)}{3} + \frac{3}{2} \pm \frac{1}{2} (2k + 1),
\]

and each of these is a valid solution to the original problem, for any positive \( k \); that is,

\[
n \geq \frac{1}{3} k(k + 1) + \frac{3}{2} \pm \frac{1}{2} (2k + 1) \geq \frac{1}{6} k(k + 1) = p,
\]

for all integers \( k \). This exhausts all possibilities for \((n, p)\). The characterization is thus: \( n = 2p; 0 \leq p \leq n < 3 \) or \((n, p) = \left( \frac{1}{3} k(k + 1) + \frac{3}{2} \pm \frac{1}{2} (2k + 1), \frac{1}{6} k(k + 1) \right) \), for some integer \( k \geq 0 \) with \( k \equiv 0 \) or \( 2 \pmod{3} \).

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1. In fact, the middle inequality is true for all real \( k \) except those strictly between 2 and 3.
2. Vacuously true.
3. If \( k = 0 \), the result is vacuously true.