11187. Proposed by Li Zhou, Polk Community College, Winter Haven, FL. Find a closed formula for the number of ways to tile a 4 by n rectangle with 1 by 2 dominoes.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: The answer can be expressed in two forms; one is

$$
\begin{pmatrix}
1 & 1 & 0 & 2 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
0 \\
1 \\
1
\end{pmatrix},
\quad (1)
$$

and the other is

$$
\left(\frac{1}{4} - \frac{\sqrt{29}}{116} + \frac{\sqrt{55}\sqrt{7-\sqrt{29}}}{116}\right) \left(\frac{1-\sqrt{29}+\sqrt{14-2\sqrt{29}}}{4}\right)^n + \left(\frac{1}{4} - \frac{\sqrt{29}}{116} + \frac{\sqrt{55}\sqrt{7+\sqrt{29}}}{116}\right) \left(\frac{1-\sqrt{29}-\sqrt{14-2\sqrt{29}}}{4}\right)^n
+ \left(\frac{1}{4} + \frac{\sqrt{29}}{116} - \frac{\sqrt{55}\sqrt{7+\sqrt{29}}}{116}\right) \left(\frac{1+\sqrt{29}+\sqrt{14-2\sqrt{29}}}{4}\right)^n + \left(\frac{1}{4} + \frac{\sqrt{29}}{116} - \frac{\sqrt{55}\sqrt{7-\sqrt{29}}}{116}\right) \left(\frac{1+\sqrt{29}-\sqrt{14-2\sqrt{29}}}{4}\right)^n.
$$

(2)

Both are obtained after finding several recurrences that solve this, and some related, problems.

In this discussion, the 4 \times n rectangle will have a width of 4 squares, and its squares will be indexed by \{1, 2, 3, 4\} \times \{1, 2, \ldots, n\}. An arch of height \(n\) will be the 4 \times n rectangle with the squares (1, \(n\)) and (4, \(n\)) removed. A tower of height \(n\) will be the 4 \times n rectangle with the squares (2, \(n\)) and (3, \(n\)) removed. A leaner of height \(n\) will be a 4 \times n rectangle with the squares (1, \(n\)) and (2, \(n\)) removed. Note that arches, towers, and leaners represent all possible ways to remove two squares of “opposite colors” from the top row of a rectangle of height \(n\), once the squares have been colored like a chessboard.

The sequences \(R_n, A_n, T_n,\) and \(L_n\) will represent, respectively, the number of ways to tile (using 1 \times 2 dominoes) a rectangle, an arch, a tower, and a leaner of height \(n\). Brute force yields the equations \(R_1 = 1, A_1 = 1, T_1 = 0, L_1 = 1\) and \(R_2 = 5\).

Now we consider \(A_n\), the number of ways to tile an arch of height \(n\). For such a tiling, the two squares (2, \(n\)) and (3, \(n\)) are either covered by the same domino, or by different dominoes. If they are covered by the same domino, the rest of the tiling tiles a rectangle of height \(n - 1\). Since the number of ways to tile a rectangle of height \(n - 1\) is \(R_{n-1}\), this is also the number of tilings of an arch of height \(n\) where the two squares at the top are covered by the same domino. If the two squares (2, \(n\)) and (3, \(n\)) are covered by two dominoes, the rest of the tiling tiles a tower of height \(n - 1\), and there are \(T_{n-1}\) of these tilings. Consequently,

\[A_n = R_{n-1} + T_{n-1},\]

for \(n \geq 2\).

Similar reasoning leads to the rest of the equations given below:

\[R_n = R_{n-2} + 2L_{n-1} + A_{n-1} + R_{n-1}\]
\[A_n = R_{n-1} + T_{n-1}\]
\[T_n = A_{n-1}\]
\[L_n = L_{n-1} + R_{n-1}\]

(3)

as long as the subscripts are all positive integers.

To get form (1) for \(R_n\), we let \(X_n = (R_n, A_n, T_n, L_n, R_{n-1})^T\); then the system (3) can be written as the matrix equation

\[X_n = \begin{pmatrix} R_n \\ A_n \\ T_n \\ L_n \\ R_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} R_{n-1} \\ A_{n-1} \\ T_{n-1} \\ L_{n-1} \\ R_{n-2} \end{pmatrix} = MX_{n-1},\]

(4)

where \(n \geq 2\), and \(M\) is the 5 \times 5 matrix in (4).
Iterating this matrix equation yields
\[ X_n = M^{n-1}X_1, \]
and to extract \( R_n \), we can multiply \( X_n \) on the left by (1 0 0 0 0). Since we found values for \( R_1, A_1, T_1, \) and \( L_1, \) and \( R_0 = 1 \) (since the “null tiling” is a tiling, and this value also continues the recursions in (3)), \( X_1 = (1 1 0 1 1)^T, \) and we obtain (1) above.

Form (2) above can be obtained by diagonalizing the matrix \( M \), which turns out to have distinct eigenvalues, and by using the fact that \( (PD^{-1})^k = PD^kP^{-1} \). It can also be obtained by manipulating the recurrences themselves, which is illustrated below.

We start with the equation \( T_n = A_{n-1} \), which allows us to remove the \( T \)’s from the equations in (3). Since this equation also implies that \( T_{n-1} = A_{n-2} \), we now have the system of recurrences
\begin{align*}
R_n &= R_{n-2} + 2L_{n-1} + A_{n-1} + R_{n-1}, \\
A_n &= R_{n-1} + A_{n-2}, \\
L_n &= L_{n-1} + R_{n-1}.
\end{align*}

Then we can eliminate the \( R \)’s with the second recurrence, which implies that \( R_{n-1} = A_n - A_{n-2} \). After this replacement, we get
\begin{align*}
2L_{n-1} &= A_{n+1} - A_n - 3A_{n-1} + A_{n-2} + A_{n-3}, \\
L_n &= L_{n-1} + A_n - A_{n-2}.
\end{align*}

Then we eliminate the \( L \)’s, using the first equation, and arrive at
\[ A_{n+2} - 2A_{n+1} - 4A_n + 4A_{n-1} + 2A_{n-2} - A_{n-3} = 0, \tag{5} \]
with suitable initial conditions.

Now we are on familiar ground, and we can use the following method to find a formula for \( A_n \); it is discussed in many introductory combinatorics books, such as Rosen’s *Discrete Mathematics and Its Applications*. The equation (5) is a homogeneous linear recurrence with constant coefficients, so we look for a solution of the form \( A_n = r^n \), and this substitution leads to the characteristic equation
\[ r^5 - 2r^4 - 4r^3 + 4r^2 + 2r - 1 = 0, \]
which has five real roots:
\[ 1, \frac{1}{2} \left( \frac{1 - \sqrt{29}}{2} \pm \sqrt{\frac{7 - \sqrt{29}}{2}} \right), \frac{1}{2} \left( \frac{1 + \sqrt{29}}{2} \pm \sqrt{\frac{7 + \sqrt{29}}{2}} \right), \]
and which will be denoted \( r_1, \ldots, r_5 \).

Not only is \( A_n = r_i^n \) a solution to (5) for any \( i \), but so is any sequence of the form
\[ C_1r_1^n + C_2r_2^n + \cdots + C_5r_5^n, \]
for arbitrary constants \( C_i \). Because \( L_n \) and \( R_n \) can be written as linear combinations of \( A_k \) for a finite number of values of \( k \), \( L_n \) and \( R_n \) will also be of this form. Once we calculate \( R_1 \) through \( R_5 \) (using the recurrence) and setting
\[ R_n = c_1r_1^n + c_2r_2^n + \cdots + c_5r_5^n, \]
for \( n = 1, \ldots, 5 \), we obtain a system of linear equations in the \( c_i \)’s which has a unique solution. Substituting these values in for the \( c_i \)’s yields (2) above. The details are left as an exercise for the overzealous reader.

This procedure (generating and solving a recurrence) has been used to show that the number of tilings of a 2 by \( n \) rectangle with 1 by 2 dominoes is a shifted Fibonacci sequence.