Given a monic polynomial $p$ of degree $n$ with complex coefficients, let $A_p$ be the $(n+1) \times (n+1)$ matrix with $p(-i+j)$ in position $(i,j)$, and let $D_p$ be the determinant of $A_p$. Show that $D_p$ depends only on $n$, and find its value in terms of $n$.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: $D_p = (n!)^{n+1}$. To see this, perform the following matrix operations on $A_p$, in this order. (Note that $C(n, k)$ is the binomial coefficient.)

1. For each $j$ from 0 to $n$ in increasing order, and for each $k$ between $j+2$ and $n+1$, subtract $C(k-1, j)$ times row $j+1$ from row $k$.
2. For each $j$ from 0 to $n$ in increasing order, and for each $k$ between $j+2$ and $n+1$, subtract $C(k-1, j)$ times column $j+1$ from column $k$.

Since subtracting a multiple of one row (or column) from another row (resp. column) does not change the determinant, the matrix obtained (which will be called $B_p$) has the same determinant as $A_p$.

After a lengthy but straightforward calculation, it can be shown that the entry in position $(i,j)$ of $B_p$ is

$$b_{i,j} = \sum_{k=0}^{i+j-2} (-1)^{k+j+1} C(i+j-2, k) p(k+1-i).$$

This expression can be written more compactly. Given a function $f$, define $\Delta f$ to be the function $(\Delta f)(x) = f(x+1) - f(x)$. Then $b_{i,j}$ is seen to be $(-1)^{i+j} \Delta^{i+j-2} p(j-i)$, where $\Delta^k$ indicates that application of the $\Delta$ operator $k$ times; this can be proven by induction on $i+j$.

It is well-known that, if $p$ is a polynomial of degree $n$, then $\Delta^np(x) = n! \cdot a_n$, where $a_n$ is the coefficient of $x^n$ in $p$ (which is equal to one here), and $\Delta^kp(x) = 0$ for all $k > n$.

This means $b_{i,j} = (-1)^{i+j} \cdot n!$ if $i+j = n+2$, i.e., if $(i,j)$ lies on the diagonal of $B_p$ connecting $(1,n+1)$ and $(n+1,1)$, and $b_{i,j} = 0$ if $i+j > n+2$. This is enough to calculate the determinant of $B_p$ (and hence of $A_p$). Swap row $k$ with row $n+2-k$, for all $k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$ to get $C_p$; then the determinant of $C_p$ is $(-1)^{\left\lfloor \frac{(n+1)/2} \right\rfloor}$ times the determinant of $B_p$.

Finally, $C_p$ is an upper-triangular matrix where $\left\lfloor \frac{n+1}{2} \right\rfloor$ entries are equal to $(-1) \cdot n!$, and the rest are equal to $n!$. Thus the determinant of $C_p$ is $(-1)^{\left\lfloor \frac{(n+1)/2} \right\rfloor} (n!)^{n+1}$. The determinants of $A_p$ and $B_p$ are then

$$\frac{(-1)^{\left\lfloor \frac{(n+1)/2} \right\rfloor} (n!)^{n+1}}{(-1)^{\left\lfloor \frac{(n+1)/2} \right\rfloor} (n!)^{n+1}} = (n!)^{n+1},$$

as claimed.