In a certain town of population $2n+1$, one knows those to whom one is known. For any set $A$ of $n$ citizens, there is some person amongst the other $n+1$ who knows everyone in $A$. Show that some citizen of the town knows all the others.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: The contrapositive to this statement (which is equivalent to the original statement) will be shown: If the town has $2n+1$ people, everyone knows those to whom they are known, and nobody knows everyone in the town, then there is a set $A$ of $n$ citizens, such that none of the other $n+1$ knows everyone in $A$.

First, everyone will receive a “color”, either red or green, in the following manner: Order the people as $P_1, P_2, \ldots, P_{2n+1}$. For each $i$ from 1 to $2n+1$, do the following: If person $P_i$ has not received a color yet, choose a person $P_{p(i)}$ such that $P_i$ does not know $P_{p(i)}$. If $P_{p(i)}$ has received a color already, assign the other color to $P_i$; otherwise, assign red to $P_i$ and green to $P_{p(i)}$.

Because of the way that the colors were assigned, for each “red” person, there is a “green” person that they do not know, and vice versa.

Due to symmetry, we may assume that there are at most $n$ red people, whom we put into set $A$. If the number of “red” people is $n-k$ for some $k>0$, choose $k$ green people and add them to $A$.

Now the set $A$ satisfies the claim: Everyone not in $A$ is green, and therefore there is a red person (whom is in $A$) whom they do not know. Hence no one not in $A$ knows everyone in $A$, as claimed.