and

\[ S \text{ functions; that is, let} \]

\[ \text{Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: We proceed using generating functions; that is, let} \]

\[ a_n = \sum_{k=0}^{n} 2^{-k} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i}, \]

and

\[ A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots \]

as a formal power series.

If we fix \( k \) for the moment, then the coefficient of \( z^m \) in \( (F_1 + F_3 z + F_5 z^2 + F_7 z^3 + \cdots)^{k+1} \) is

\[ \sum_{x \in S[k,m+k]} \prod_{i=1}^{k+1} F_{1+2x_i}; \]

if the expression above is multiplied out, it will consist of the sum of a bunch of terms of the form

\[ (F_{2y_1+1} z^{y_1})(F_{2y_2+1} z^{y_2}) \cdots (F_{2y_k+1} z^{y_k+1}), \]

where \( y_i \) is the power of \( z \) taken from the \( i \)th factor. In order to end up with an exponent of \( z^{m+k} \), it is necessary and sufficient to have \( y_1 + y_2 + \cdots + y_{k+1} = m \). Since \( y_i \geq 0 \) as well, this means \( y \) is one of the elements of \( S[k,k+m]\). When we include other terms, we get the rest of the elements of \( S[k,k+m]\).

Next, the coefficient of \( z^m \) needs to be transferred to a coefficient of \( z^{m+k} \), which can be done by multiplying by \( z^k \). Simultaneously, we multiply this coefficient by \( 2^{-k} \). What we have shown is that the generating function of

\[ 2^{-k} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i}, \]

is \( 2^{-k} z^k (F_1 + F_3 z + F_5 z^2 + F_7 z^3 + \cdots)^{k+1} \). Now, we sum over all \( k \) to get the generating function for \( a_n \):

\[ A(z) = \sum_{k=0}^{\infty} 2^{-k} z^k (F_1 + F_3 z + F_5 z^2 + F_7 z^3 + \cdots)^{k+1}. \]

Next, we look for a closed form for \( A(z) \). It is well-known that the generating function for all of the Fibonacci numbers is

\[ \Phi_{all}(z) = F_0 + F_1 z + F_2 z^2 + \cdots = \frac{z}{1 - z - z^2}, \]

and since \( F_0 = 0 \), we can divide both sides of this equation by \( z \) to get

\[ F_1 + F_2 z + F_3 z^2 + \cdots = \frac{1}{1 - z - z^2}, \]

and

\[ F_1 + F_3 z^2 + F_5 z^4 + \cdots = \frac{1}{2} \left[ (F_1 + F_2 z + F_3 z^2 + F_4 z^3 + \cdots) + (F_1 - F_2 z + F_3 z^2 - F_4 z^3 + \cdots) \right] \]

\[ = \frac{1}{2} \left( \frac{1}{1 - z - z^2} + \frac{1}{1 + z - z^2} \right) = \frac{1 - z^2}{1 - 3z^2 + z^4}. \]
Replacing $z$ with $\sqrt{z}$ yields

$$\Phi_{\text{odd}}(z) \equiv F_1 + F_3 z + F_5 z^2 + \cdots = \frac{1 - z}{1 - 3z + z^2}.$$ 

Now $A(z)$ is a geometric series, so $A(z)$ is

$$\sum_{k=0}^{\infty} 2^{-k} z^k (\Phi(z))^{k+1} = \frac{\Phi(z)}{1 - \frac{z}{2} \cdot \Phi(z)} = \frac{2 - 2z}{2 - 7z + 3z^2} = \frac{2/5}{2 - z} + \frac{4/5}{1 - 3z} = \frac{1/5}{1 - z/2} + \frac{4/5}{1 - 3z}.$$

Since the generating function of $r^n$ is $\frac{1}{1 - rz}$, this implies that $a_n = \frac{1}{5} \left( \frac{1}{2} \right)^n + \frac{4}{5} 3^n$, and the original expression equals $2^n$ times this, or $\frac{1}{5} (1 + 4 \cdot 6^n)$.

A natural generalization of this problem is that of replacing $2$ by some real number $\alpha$. The hardest part of the proof is finding the partial fraction decomposition of $A(z)$ above; it is nice when the denominator factors. If $2$ is replaced with $\alpha$, and the procedure above is followed, the denominator of $A(z)$ turns out to be

$$(\alpha + 1)z^2 + (-3\alpha - 1)z + \alpha.$$ 

(The special case where $\alpha = -1$ has to be dealt with separately, of course.) This quadratic factors iff its discriminant is a perfect square; that is we need

$$(-3\alpha - 1)^2 - 4\alpha(\alpha + 1) = M^2.$$ 

If we solve this quadratic for $\alpha$ in terms of $M$, we get $\alpha = \frac{-1 \pm \sqrt{5M^2 - 4}}{5}$. In the “nice” cases, $\alpha$ is a rational number and $5M^2 - 4$ is a perfect square.

The nonnegative integral solutions to the equation $5M^2 - 4 = N^2$ are well known; it turns out that $M = F_m$ (the $m$th Fibonacci number) and $N = L_m$ (the $m$th Lucas number*), for some nonnegative odd integer $m$.

The Fibonacci numbers and Lucas numbers get involved with the solution at this point. The derivation is straightforward but messy. The result (checked with Maple) is as follows:

**Proposition 1.** Let $F_{\alpha,\text{odd}}(n) = \sum_{k=0}^{n} \alpha^{-k} \sum_{x \in \mathbb{Z}[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i}$. Then, for all $n > 0$,

(a) $F_{-1,\text{odd}}(n) = -\frac{1}{2} (-2)^n$;

(b) if $\alpha = \frac{-1 + L_m}{5}$, where $m$ is an odd integer, then

$$F_{\alpha,\text{odd}}(n) = \frac{(6 - L_m - 5F_m)(-2 + 3L_m + 5F_m)}{20F_m(L_m + 4)} \cdot \frac{(2(L_m - 1)(L_m + 4))}{(5(2 + 3L_m + 5F_m))} \cdot \frac{(2L_m - 1)(L_m + 4)}{5(2 + 3L_m + 5F_m)}; \quad \text{and}$$

(c) if $\alpha = \frac{-1 - L_m}{5}$, where $m$ is an odd integer, and $\alpha \neq -1$,

$$F_{\alpha,\text{odd}}(n) = \frac{(6 + L_m + 5F_m)(-2 + 3L_m - 5F_m)}{20F_m(L_m - 4)} \cdot \frac{(2L_m + 1)(L_m - 4)}{5(2 - 3L_m - 5F_m)} \cdot \frac{(2L_m + 1)(L_m - 4)}{5(2 - 3L_m + 5F_m)}.$$ 

Note that the original problem is a special case of Proposition 1(b), where $m = 5$.

What if the even terms of the Fibonacci numbers are used in the original problem? Then the following hold (provided there are no typos; the mathematics was verified using Maple).

* The Lucas numbers satisfy the Fibonacci relation $L_n = L_{n-1} + L_{n-2}$, but start off differently: $L_1 = 1$ and $L_2 = 3$. 

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Proposition 3. Let $F_{\alpha,\text{even}}(n) = \sum_{k=0}^{n} \alpha^{n-k} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} F_{2x_i}$; then, for all $n > 0$,

(a) $F_{1,\text{even}}(n) = 3^{n-1}$ and $F_{-1,\text{even}}(n) = (-2)^n - (-1)^n$;

(b) if $\alpha = \frac{-2 + L_m}{5}$, where $m$ is an even integer, then

$$F_{\alpha,\text{even}}(n) = \frac{(L_m + 8 + 5F_m)(8 + L_m - 5F_m)(L_m - 2)}{20F_m(L_m + 3)(L_m - 7)} \cdot \left(\frac{2(L_m - 2)(L_m - 7)}{5(-6 + 3L_m + 5F_m)}\right)^n;$$

(c) if $\alpha = \frac{-2 - L_m}{5}$, and $\alpha \neq -1$, where $m$ is an even integer, then

$$F_{\alpha,\text{even}}(n) = \frac{(8 - L_m - 5F_m)(8 - L_m + 5F_m)(L_m + 2)}{20F_m(L_m - 3)(L_m + 7)} \cdot \left(\frac{2(L_m + 2)(L_m + 7)}{5(-6 - 3L_m + 5F_m)}\right)^n.$$

If the full Fibonacci sequence is used (that is, $F_x$ is substituted for $F_{2x+1}$)

Proposition 3. Let $F_{\alpha,\text{all}}(n) = \sum_{k=0}^{n} \alpha^{n-k} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} F_{x_i}$; then, for all $n > 0$,

(a) $F_{-1,\text{all}}(n) = (-1)^n$;

(b) if $\alpha = \frac{-2 + L_m}{5}$, where $m$ is an even integer, then

$$F_{\alpha,\text{all}}(n) = \frac{(L_m - 2)(4 + 3L_m + 5F_m)(4 + 3L_m - 5F_m)}{20F_m(L_m + 3)^2} \cdot \left(\frac{2(L_m - 2)(L_m + 3)}{5(2 - L_m + 5F_m)}\right)^n;$$

(c) if $\alpha = \frac{-2 - L_m}{5}$, where $m$ is an even integer, and $\alpha \neq -1$, then

$$F_{\alpha,\text{all}}(n) = \frac{(L_m + 2)(4 - 3L_m - 5F_m)(4 - 3L_m + 5F_m)}{20F_m(L_m - 3)^2} \cdot \left(\frac{2(L_m + 2)(L_m - 3)}{5(2 + L_m - 5F_m)}\right)^n.$$