11218. Proposed by Gary Gordon, Lafayette College, Easton, PA. Consider the following algorithm, which takes as input a positive integer $n$ and proceeds by rounds, listing in each round certain positive integers between 1 and $n$ inclusive, ultimately producing as output a positive integer $f(n)$, the last number to be listed. In the 0th round, list 1. In the first round, list, in increasing order, all primes less than $n$. In the second round, list in increasing order all numbers that have not yet been listed and are of the form $2p$, where $p$ is prime. Continue in this fashion, listing numbers of the form $3p$, $4p$, and so on until all numbers between 1 and $n$ have been listed. Thus $f(10) = 8$ because the list eventually reaches the state $\{1, 2, 3, 5, 7, 4, 6, 10, 9, 8\}$, while $f(20) = 16$ and $f(30) = 27$.

(a) Find $f(2006)$.
(b) Describe the range of $f$.
(c) Find $\liminf_{n \to +\infty} \frac{f(n)}{n}$ and $\limsup_{n \to +\infty} \frac{f(n)}{n}$.

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: We proceed by establishing several intermediary results about the function $f$, using some extra functions. These extra functions have bizarre behavior when $n = 1$, so we will be concentrating on $n \geq 2$.

The first of these will be $R(n)$, which is the first round in which $n$ appears; then $r(n)$ will be defined to be $R(f(n))$. Note that if $n \geq 2$ appears in round $m$, then $n = m \cdot p$, where $p$ is a prime. To find the first round in which $n$ appears, we need to make $p$ be as large as possible and simultaneously remain a divisor of $n$. Thus, to minimize $m$, we need to let $p$ be the greatest prime factor of $n$, which will be denoted by GPF $(n)$. Then we will have $R(n) = \frac{n}{\text{GPF}(n)}$, for $n \geq 2$. Note that $f(n) = R(k) \cdot \text{GPF}(k)$, where $k = f(n)$.

Note that the last number in $\{1, \ldots, n\}$ to be listed will be one which maximizes $R(k)$ among the integers $k$ between 1 and $n$; thus $r(n) = \max_{2 \leq k \leq n} R(k)$, when $n \geq 2$. Now we present some “obvious” facts about $f$ and $r$:

**Lemma 1.** If $n \geq 1$ is an integer, then:

(a) $f$ is non-decreasing;
(b) $r$ is non-decreasing;
(c) $f(n) \leq n$;
(d) $n$ is in the range of $f$ iff $n = f(n)$;
(e) $f(n) = m$ iff for all $k$ between 1 and $n$: $R(m) \geq R(k)$, and if $R(m) = R(k)$, then $m \geq k$;
(f) $n$ is in the range of $f$ iff $R(n) \geq R(k)$ for all $k$ between 1 and $n$;
(g) If $n \geq 2$, $R(2n) = 2R(n)$.

To prove Lemma 1(d), note that the “if” part is trivial; now suppose $n = f(k)$ for some $k \geq 1$. If $k > n$, then the elements in $\{1, \ldots, n - 1\} \cup \{n + 1, \ldots, k\}$ are listed before $n$ in the algorithm described above. But then just the elements $\{1, \ldots, n - 1\}$ are all listed before $n$, which implies that $f(n) = n$, which is what we wanted to show.

Part (e) is by definition, and part (f) follows from parts (d) and (e).

Part (g) follows since GPF $(n) = \text{GPF}(2n)$, when $n \geq 2$.

Now we move on to some non-obvious properties of $f$:

**Lemma 2.** If $n \geq 2$ is in the range of $f$, then GPF $(n) \leq 3$. Consequently, $n = 2^a \cdot 3^b$, for some nonnegative integers $a$, $b$.

**Proof:** Suppose $n \geq 2$ is in the range of $f$. Choose $m$ to be the largest power of 2 less than $n$, so that $m < n$ and $n \leq 2m$. Then

$$\frac{n}{4} \leq \frac{m}{2} = R(m) \leq r(m) \leq r(n) = R(n) = \frac{n}{\text{GPF}(n)},$$

due to the form of $m$, the definition of $r(m)$, Lemma 1(b), the assumption that $f(n) = n$, and the definition of $R$. But this implies GPF $(n) \leq 4$, which is equivalent to what we wanted to prove. This proves Lemma 2. ■

However, not every number of the form $2^a \cdot 3^b$ is in the range of $f$; in particular, $R(9) = 3 < 4 = R(8)$, so by Lemma 1(f), 9 is not in the range of $f$. The question of whether $f(2^a \cdot 3^b) = 2^a \cdot 3^b$ will be answered in stages, starting with:
Lemma 3. \( n \) is in the range of \( f \) iff \( 2n \) is in the range of \( f \).

Proof: Lemma 3 is true if \( n = 1 \), so we assume that \( n \geq 2 \). We start by proving the “if” part: If \( 2n \) is in the range of \( f \), then Lemma 1(f) implies
\[
R(k) \leq R(2n), \quad \forall k \in \{2, \ldots, 2n\}.
\]

Now choose an \( i \) between 2 and \( n \). Then, from (\(*\)) and Lemma 1(g), we deduce
\[
2R(i) = R(2i) \leq R(2n) = 2R(n).
\]

This inequality implies \( R(i) \leq R(n) \) whenever \( 2 \leq i \leq n \). Thus \( n \) is in the range of \( f \), by Lemma 1(f).

Now for the “only if” part. We assume that \( R(k) \leq R(n) \), for all \( k \in \{2, \ldots, n\} \), and need to show (\(*\)). Thus, let \( i \in \{2, \ldots, 2n\} \).

If \( i \) is not a power of 2, then \( \text{GPF}(i) \geq 3 \), and
\[
R(i) = \frac{i}{\text{GPF}(i)} \leq \frac{2n}{\text{GPF}(i)} \leq \frac{2n}{3} \leq \frac{2n}{\text{GPF}(2n)} = R(n).
\]

If \( i = 2 \), \( R(i) = 1 \leq R(n) \), as \( n > 1 \). Lastly, if \( i \) is any other power of 2 between 2 and \( 2n \),
\[
R(i) = 2R\left(\frac{i}{2}\right) \leq 2R(n) = R(2n),
\]

which is the last part needed to show (\(*\)). Consequently, \( 2n \) is in the range of \( f \), finishing the proof of Lemma 3.

Before determining which values of \( k \) have \( f(3^k) = 3^k \), we remark that Lemma 2 states that we only need to examine numbers of the form \( 2^a \cdot 3^b \) to determine \( f(n) \); in fact, \( f(n) \) will be the maximum \( f \) value of these numbers. Lemma 3 states even more; we only need to consider numbers of the form \( 2^a \cdot 3^b \) where \( 2^a + 3^b > n \), since any number of the form \( 2^a \cdot 3^b \) will be eliminated before \( 2^a \cdot 3^c \) if \( c < a \). Then we only need to determine what order these numbers are eliminated, and choose the one eliminated last. We have thus shown the following, where \( L(i) = \lfloor \frac{n}{3} \rfloor = \lfloor \log n - i \log 3 \rfloor \), and \( \log n \) denotes the logarithm of \( n \) base 2:

Lemma 4. The following algorithm calculates \( f(n) \), for \( n \geq 2 \); furthermore, this algorithm is polynomial time in \( \log n \), and also the number of bits necessary to express \( n \).

(a) Calculate \( a_0 = 2L(0) - 1 \) and \( b_0 = 2L(0) \);
(b) Calculate \( a_i = 2L(i) \cdot 3^{i-1} \) and \( b_i = 2L(i) \cdot 3^i \), for \( i = 1, \ldots, \lfloor \log_3 n \rfloor \);
(c) Determine the value \( j \) between 0 and \( \lfloor \log_3 n \rfloor \) such that \( a_j \geq a_i \), and if equality holds, \( b_j > b_i \);
(d) Return \( b_j \).

Now we can determine when \( f(3^k) = 3^k \):

Lemma 5. \( f(3^k) = 3^k \) iff \( k \leq 1 \) or \( (k - 1) \log 3 > \lfloor k \log 3 \rfloor - 1 \).

Proof: The result holds if \( k \leq 1 \), so suppose \( k \geq 2 \). Using the notation of Lemma 4, we want to show that \( a_k > a_i \) for all \( 0 \leq i < k \).

First, we will concentrate on the case where \( i > 0 \); \( a_k > a_i \) iff the following inequalities also hold, which are all equivalent by the properties of logarithms:
\[
2^{\lfloor \log 3^{k-1} \log 3 \rfloor} \cdot 3^{k-1} > 2^{\lfloor \log 3^{i-1} \log 3 \rfloor} \cdot 3^{i-1} = 2^{(k-1) \log 3} \cdot 3^{k-1} \geq 2^{(k-i) \log 3} \cdot 3^i
\]
\[
3^{k-i} > 2^{(k-i) \log 3}
\]
\[
(k - i) \log 3 > \lfloor (k - i) \log 3 \rfloor
\]
Since the last inequality is true when \( k - i \) is replaced by an arbitrary positive integer (and \( \log 3 \) is irrational), this proves the intermediate claim.

Now we can state: \( f(3^k) = 3^k \) iff \( a_k > a_0 \). (Note that we cannot have \( a_k = a_0 \), because 3 divides \( a_k \) but not \( a_0 \).) But the inequality \( a_k > a_0 \) is equivalent to the inequality

\[
3^{k-1} > 2^{|k \log 3|-1},
\]

which, after some algebra, is equivalent to \( (k - 1) \log 3 > |k \log 3| - 1 \).

Now we turn to the questions which were asked.

(a) \( f(2006) = 1944 = 2^3 \cdot 3^4 \). Note that 1944 is the largest integer \( \leq 2006 \) of the form \( 2^a \cdot 3^b \), and that \( f(3^2) = 3^2 \) by Lemma 5. (This author wonders whether the year 1944 has any significance for the proposer.)

To show how much better the algorithm in Lemma 4 is than the original, Maple was used on an Intel Celeron processor (running at 1.4 GHz) to determine that \( f(10^{100}) = 2^{332} \). The calculation took two seconds and 4.25M of memory.

(b) Combining Lemmas 3 and 5 implies that the range of \( f \) is

\[
\left\{ 2^a \cdot 3^b : a, b \geq 0, \quad a, b \in \mathbb{Z}, \quad \text{and} \quad (b \leq 1 \text{ or } (k - 1) \log 3 > |k \log 3| - 1) \right\}.
\]

(c) Lemma 1(c) implies that \( \limsup_{n \to \infty} \frac{f(n)}{n} \leq 1 \), and since there are an infinite number of integers \( n \) such that

\[
\frac{f(n)}{n} = 1, \quad \limsup_{n \to \infty} \frac{f(n)}{n} = 1.
\]

We proceed in a similar way to find \( \liminf_{n \to \infty} \frac{f(n)}{n} \). First of all, Lemma 2(a), Lemma 3, and the fact that \( f(3) = 3 \) imply that \( f(2^k) = 2^k \) and \( f(3 \cdot 2^k) = 3 \cdot 2^k \).

If \( 2 \cdot 2^k \leq n \leq 3 \cdot 2^k \) for some integer \( k \), then (since \( f \) is non-decreasing) \( \frac{f(n)}{n} \geq \frac{f(2^{k+1})}{3 \cdot 2^k} = \frac{2^{k+1}}{3 \cdot 2^k} = \frac{2}{3} \), and if \( 3 \cdot 2^k \leq n \leq 4 \cdot 2^k \), \( \frac{f(n)}{n} \geq \frac{f(3 \cdot 2^k)}{4 \cdot 2^k} = \frac{3}{4} \). Thus, since \( \frac{f(n)}{n} \geq \frac{2}{3} \) for all \( n \geq 2 \), \( \liminf_{n \to \infty} \frac{f(n)}{n} \geq \frac{2}{3} \).

Now we consider the (strictly) increasing sequence of positive integers \( 3 \cdot 2^k - 1 \). If we can show that \( f(3 \cdot 2^k - 1) = 2 \cdot 2^k \), then we can deduce

\[
\liminf_{n \to \infty} \frac{f(n)}{n} \leq \lim_{k \to \infty} \frac{2 \cdot 2^k}{3 \cdot 2^k - 1} = \frac{2}{3},
\]

and we will find that \( \liminf_{n \to \infty} \frac{f(n)}{n} = \frac{2}{3} \).

To determine \( f(3 \cdot 2^k - 1) \), note that \( r(2 \cdot 2^k) = 2^k = r(3 \cdot 2^k) \). Since \( r \) is nondecreasing (by Lemma 1(b)), \( r(3 \cdot 2^k - 1) = 2^k \) as well. Then (if we define \( p(n) \) to be GPF \( f(n) \)):

\[
2^k p(3 \cdot 2^k - 1) = f(3 \cdot 2^k - 1) \geq f(2 \cdot 2^k) = 2 \cdot 2^k, \quad \text{and}
\]

\[
2^k p(3 \cdot 2^k - 1) = f(3 \cdot 2^k - 1) \leq 3 \cdot 2^k - 1 < 3 \cdot 2^k,
\]

which together imply \( 2 \leq p(3 \cdot 2^k - 1) < 3 \), so \( p(3 \cdot 2^k - 1) = 2 \) and \( f(3 \cdot 2^k - 1) = p(3 \cdot 2^k - 1) r(3 \cdot 2^k - 1) = 2 \cdot 2^k \), as desired.