Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ:

**Theorem 1.** The chromatic number of $M_n$ is

$$\mu_n = \left\lfloor \frac{7 + \sqrt{134064 \cdot 20^n + 242592 \cdot 8^n - 358967}}{2} \right\rfloor.$$ 

This solution requires the use of a published result:

**Theorem 2.** (Ringel, Youngs) The chromatic number of the orientable surface of genus $g$ is

$$\left(\frac{7 + \sqrt{48g + 1}}{2}\right).$$


Since the Euler characteristic of an orientable surface of genus $g$ is $2 - 2g$, it suffices to find the Euler characteristic of $M_n$ (which will be denoted $\mu_n$; the Euler characteristic of an arbitrary set $A$ will be denoted $\chi(A)$). We will need another result, about the Euler characteristic of the Sierpinski Gasket.

**Definition.** Let the stage-$n$ Sierpinski Gasket be $S_n$, defined as follows: $S_0$ is defined to be the unit square, and given $S_{n-1}$, construct $S_n$ by drilling each of the $8^{n-1}$ sub-squares of edge-length $3^{1-n}$ in $S_{n-1}$. The Euler characteristic of $S_n$ will be denoted $\gamma_n$.

**Lemma 3.** The Euler characteristic of $S_n$ is $\gamma_n = \frac{16}{7} - \frac{2}{7} \cdot 8^n$.

**Proof of Lemma 3.** The genus of $S_n$ is equal to the number of “holes” in $S_n$, which is $1 + 8 + 8^2 + \ldots + 8^{n-1} = \frac{8^n - 1}{7}$. The Euler characteristic of $S_n$ is thus $2 - 2 \cdot \frac{8^n - 1}{7} = \frac{16}{7} - \frac{2}{7} \cdot 8^n$.

**Lemma 4.** The Euler characteristic of $M_n$, $\mu_n$, satisfies the recurrence:

$$\begin{align*}
\mu_0 &= 2, \\
\mu_{n+1} &= 20 \mu_n - \frac{384}{7} + \frac{48}{7} \cdot 8^n, \quad n \geq 0.
\end{align*}$$

(\ast)

**Proof of Lemma 4.** Clearly $\mu_0 = 2$. Now suppose $n \geq 1$.

We will view $M_{n+1}$ as 20 copies of $M_n$, consisting of eight “corner cubes” (each adjacent to three other copies of $M_n$) and twelve “edge cubes” (each adjacent to two other copies of $M_n$). Fix $n$, and let $N_0$ consist of the eight corner cubes of $M_{n+1}$, and for each $k$ between 1 and 12, let $N_k$ be the union of $N_{k-1}$ and one of the edge cubes not in $N_{k-1}$. (It does not matter which one in particular.) Thus $N_{12} = M_{n+1}$.

Since $N_0$ is the union of eight pairwise disjoint copies of $M_n$, the Euler characteristic of $N_0$ is $8 \cdot \mu_n$. Now pick a $k$ between 1 and 12. $N_k$ is the union of $N_{k-1}$ and a copy of $M_n$. The Euler characteristic satisfies the following inclusion-exclusion equality:

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Since the intersection of $N_{k-1}$ and the new copy of $M_n$ is two disjoint copies of $S_n$,

$$\chi(N_k) = \chi(N_{k-1} \cup M_n) = \chi(N_{k-1}) + \chi(M_n) - \chi(N_{k-1} \cap M_n) = \chi(N_{k-1}) + \mu_n - \chi(2S_n) = \chi(N_{k-1}) + \mu_n - 2 \gamma_n.$$ 

Repeated iteration of this equality, with an application of Lemma 3, yields

$$\mu_{n+1} = \chi(M_{n+1}) = \chi(N_{12}) = \chi(N_0) + 12 \mu_n - 24 \gamma_n = 8 \mu_n + 12 \mu_n - 24 \left(\frac{16}{7} - \frac{2}{7} \cdot 8^n\right),$$

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which is equivalent to what we wanted to show.

Now we can prove Theorem 1. First, we solve the recurrence given by Lemma 4. This is a linear non-homogeneous recurrence with constant coefficients, and it can be solved using methods found in any introductory combinatorics book, such as Rosen’s *Discrete Mathematics and Its Applications*.

Briefly, we first solve the homogeneous recurrence $h_{n+1} = 20h_n$ (the recurrence $\ast$, without the terms not involving $\mu$); it is easy to see that a solution is of the form $h_n = C_1 \cdot 20^n$. Then we look for one particular solution to the recurrence

$$p_{n+1} = 20p_n - \frac{384}{7} + \frac{48}{7} \cdot 8^n,$$

by looking for a solution of the form

$$p_n = C_2 \cdot 1^n + C_3 \cdot 8^n.$$

It turns out that $C_2 = \frac{384}{133}$ and $C_3 = -\frac{4}{7}$ makes $p_n$ satisfy $\ast$, so

$$p_n = \frac{384}{133} - \frac{4}{7} \cdot 8^n.$$

Then any solution to $\ast$ is of the form

$$\mu_n = h_n + p_n = C_1 \cdot 20^n + \frac{384}{133} - \frac{4}{7} \cdot 8^n,$$

and all we need to do is to choose $C_1$ so that $\mu_0 = 2$. To have this, we must have $C_1 = -\frac{6}{19}$, so

$$\mu_n = -\frac{6}{19} \cdot 20^n + \frac{384}{133} - \frac{4}{7} \cdot 8^n.$$

Now we substitute $g = \frac{2 - \mu_n}{2}$ into the formula given in Theorem 2, and obtain Theorem 1 after simplifying.

The solver proposes the following conjecture:

**Conjecture.** This problem was inspired by the confusion between the notations for chromatic number and Euler characteristic.