(A1) Let $f$ be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points $P$ in the plane?

(A2) Functions $f$, $g$, $h$ are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$
f' = 2f^2g + \frac{1}{gh}, \quad f(0) = 1, \quad g' = fg^2h + \frac{4}{fh}, \quad g(0) = 1, \quad h' = 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1.\$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

(A3) Let $d_3$ be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example, $d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$. The argument of $\cos$ is always in radians, not degrees.) Evaluate $\lim_{n \to \infty} d_n$.

(A4) Let $S$ be a set of rational numbers such that

(a) $0 \in S$;
(b) If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$; and
(c) If $x \in S$ and $x \not\in \{0, 1\}$, then $\frac{1}{x(x-1)} \in S$.

Must $S$ contain all rational numbers?

(A5) Is there a finite abelian group $G$ such that the product of the orders of all its elements is $2^{2009}$?

(A6) Let $f : [0,1]^2 \to R$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0,y) \, dy$, $b = \int_0^1 f(1,y) \, dy$, $c = \int_0^1 f(x,1) \, dx$, and $d = \int_0^1 f(x,1) \, dx$. Prove or disprove: There must be a point $(x_0, y_0)$ in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$
(B1) Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example, \( \frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3!} \).

(B2) A game involves jumping to the right on the real number line. If \( a \) and \( b \) are real numbers and \( b > a \), the cost of jumping from \( a \) to \( b \) is \( b^3 - ab^2 \). For what real numbers \( c \) can one travel from 0 to 1 in a finite number of jumps with total cost exactly \( c \)?

(B3) Call a subset \( S \) of \( \{1, 2, \ldots, n\} \) mediocre if it has the following property: Whenever \( a \) and \( b \) are elements of \( S \) whose average is an integer, that average is also an element of \( S \). Let \( A(n) \) be the number of mediocre subsets of \( \{1, 2, \ldots, n\} \). [For instance, every subset of \( \{1, 2, 3\} \) except \( \{1, 3\} \) is mediocre, so \( A(3) = 7 \).] Find all positive integers \( n \) such that \( A(n + 2) - 2A(n + 1) + A(n) = 1 \).

(B4) Say that a polynomial with real coefficients in two variables, \( x, y \), is balanced if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space \( V \) over \( R \). Find the dimension of \( V \).

(B5) Let \( f : (1, \infty) \to R \) be a differentiable function such that
\[
f'(x) = \frac{x^2 - (f(x))^2}{x^2((f(x))^2 + 1)} \quad \text{for all } x > 1.
\]
Prove that \( \lim_{x \to \infty} f(x) = \infty \).

(B6) Prove that for every positive integer \( n \), there is a sequence of integers \( a_0, a_1, \ldots, a_{2009} \) with \( a_0 = 0 \) and \( a_{2009} = n \) such that each term after \( a_0 \) is either an earlier term plus \( 2^k \) for some nonnegative integer \( k \), or of the form \( b \mod c \) for some earlier positive terms \( b \) and \( c \). [Here \( b \mod c \) denotes the remainder when \( b \) is divided by \( c \), so \( 0 \leq (b \mod c) < c \).]