ALL MINIMUM $C_5$-SATURATED GRAPHS

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Abstract. A graph is $C_5$-saturated if it has no five-cycle as a subgraph, but does contain a $C_5$ after the addition of any new edge. Extending our previous result, we prove that the minimum number of edges in a $C_5$-saturated graph on $n$ vertices is $sat(n, C_5) = [10(n - 1)/7] - 1$ for $11 \leq n \leq 14$, or $n = 16, 18, 20$, and is $[10(n - 1)/7]$ for all other $n \geq 5$, and we also prove that the only $C_5$-saturated graphs with $sat(n, C_5)$ edges are the graphs described in Section 2.

§ 1. Introduction.

All graphs studied are simple ones. We denote a path, a cycle, a star, a complete graph, and a complete $r$-uniform hypergraph with $n$ vertices by $P_n$, $C_n$, $S_n$, $K_n$, and $K^r_n$, respectively. The family of $r$-uniform hypergraphs on $n$ vertices is denoted by $\mathcal{H}_r(t, m)$. For a graph $G$, let $N(x)$ of $x \in V(G)$ be the set of the vertices adjacent to $x$, and $d(x) = |N(x)|$, $n(G) = |V(G)|$, $e(G) = |E(G)|$, and $\delta(G) = \min\{d(x) : x \in V(G)\}$. We denote a path $x_1, ..., x_k$ by $P(x_1, ..., x_k)$. If $A \subseteq V(G)$, we define $G[A]$ to be the subgraph with vertex set $A$ and edge set, denoted by $E(G[A])$, $\{xy \in E(G) : x, y \in A\}$. We write $e(G[A])$ as $e(A)$.

Let $\mathcal{F}$ be a family of graphs or hypergraphs. A hypergraph is $\mathcal{F}$-saturated if it has no $F \in \mathcal{F}$ as a subhypergraph, but does contain some $H \in \mathcal{F}$ after the addition of any new edge. A hypergraph $G$ is weakly $\mathcal{F}$-saturated if the edges not in $G$ can be added one by one in some order such that a new subhypergraph isomorphic to some $F \in \mathcal{F}$ appears every time a new edge is added. The minimum and maximum number of edges in an $\mathcal{F}$-saturated graph on $n$ vertices are denoted by $sat(n, \mathcal{F})$ and $ex(n, \mathcal{F})$, respectively. The minimum number of edges in a weakly $\mathcal{F}$-saturated graph on $n$ vertices is denoted by $w-sat(n, \mathcal{F})$. An $\mathcal{F}$-saturated graph $G$ on $n$ vertices with $e(G) = sat(n, \mathcal{F})$ is called a $sat(n, \mathcal{F})$-graph. The problem of determining $ex(n, \mathcal{F})$ is the famous Turán’s problem. If $\mathcal{F} = \{F\}$, we also write $sat(n, \mathcal{F})$ as $sat(n, F)$. Erdős, Hajnal, and Moon [7] proved that the $sat(n, K_k)$-graph is obtained by joining each of the $n - k + 2$ independent vertices to every vertex in a $K_{k-2}$. Kászonyi and Tuza [12] determined $sat(n, F)$ for $F = S_k, K_2, P_k$, and also proved that $sat(n, \mathcal{F}) = O(n)$ for any family $\mathcal{F}$ of graphs.


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hypergraphs \( F \) with few edges and \( w\)-sat\((n, \mathcal{H}_r(r + 1, 3)) \). Pikhurko [14] proved Tuza’s conjecture that \( \text{sat}(n, \mathcal{F}) = O(n^{-r-1}) \) for all families of \( r \)-uniform hypergraphs whose independence numbers are bounded by a constant. Pikhurko [15] also proved Tuza’s other conjecture [18] for \( w\)-sat\((n, \mathcal{H}_r(r + 1, m)) \). For more results and open problems, see [16].

For \( \text{ex}(n, C_k) \), Bollobás [4, Chapter 3, Corollary 5.4] proved that \( \text{ex}(n, C_k) = \lceil n^{2}/4 \rceil \) for odd \( k \) with \( k \leq (n+3)/2 \). For a rare exact result on \( \text{ex}(n, C_k) \), see Füredi’s papers on \( \text{ex}(n, C_4) [9, 10] \). The problem of determining \( \text{sat}(n, C_k) \) is also mentioned in Bollobás’ Extremal Graph Theory [4]. Ollmann [13] pointed out that the \( \text{sat}(n, C_3) \)-graph is the star \( S_n \), and he obtained all \( \text{sat}(n, C_4) \)-graphs. Later Tuza [19] gave a shorter proof for \( \text{sat}(n, C_4) \). Ashkenazi [1] described the properties of \( C_3 \)-saturated graphs, planar \( C_3 \)-saturated graphs, and \( C_4 \)-free \( C_3 \)-saturated graphs. Fisher, Fraughnaugh, and Langley [8] constructed the \( C_5 \)-saturated graphs \( H_{7k+1} \) with \( \lceil 10(n-1)/7 \rceil \) edges and the ones in \( G_1 \) with \( \lceil 10(n-1)/7 \rceil - 1 \) edges defined in Section 2. Barefoot, Clark, Entringer, Porter, Székele, and Tuza [2] gave new upper bounds for \( \text{sat}(n, C_k) \) for \( k = 6, 7, 12 \), and showed that \( n + c_1 n/k \leq \text{sat}(n, C_k) \leq n + c_2 n/k \) for some positive \( c_1, c_2 \). Recently, Gould, Luczak, and Schmitt [11] improved the value of \( c_2 \) and upper bounds for \( \text{sat}(n, C_k) \) for \( k = 8, 9, 11, 13, 15 \). Barefoot, Clark, Entringer, Porter, Székely, and Tuza [2] also suggested that \( \text{sat}(n, C_k) \)-graphs have the structure of identifying a vertex in several copies of \( C_k \)-builders. In [5] we proved that \( \text{sat}(n, C_5) \) satisfies (1.1) below for \( n \geq 11 \). Note that the \( \text{sat}(n, C_5) \)-graph for \( n < 5 \) is the complete graph \( K_n \). Now we show that \( \text{sat}(n, C_5) \) satisfies (1.1) for all \( n \geq 5 \), and the only \( \text{sat}(n, C_5) \)-graphs are those in Figure 1 or in \( G_1, G_2, G_3 \) defined in Section 2. This confirms that the \( \text{sat}(n, C_5) \)-graphs for most values of \( n \) have the structure of identifying a vertex in several copies of \( C_5 \)-builders, \( G_3^8 \) in Figure 1.

**Theorem 1.**

\[
\text{sat}(n, C_5) = \begin{cases} 
\lceil 10(n-1)/7 \rceil - 1 & \text{if } n \in N_0 = \{11, 12, 13, 14, 16, 18, 20\}, \\
\lceil 10(n-1)/7 \rceil & \text{if } n \notin N_0 \text{ and } n \geq 5.
\end{cases}
\]  

(1.1)

The set of \( \text{sat}(n, C_5) \)-graphs with \( n \geq 5 \) consists of the graphs in Figure 1 or in \( G_1 \cup G_2 \cup G_3 \) defined in Section 2.

We present \( \text{sat}(n, C_5) \)-graphs in Section 2, recall the properties and structures of \( C_5 \)-saturated graphs obtained in [5] in Section 3, obtain a lower bound for \( \text{sat}(n, C_5) \)-graphs \( G \) with \( \delta(G) = 2 \) in Section 4, discuss the \( \text{sat}(n, C_5) \)-graphs \( G \) with \( \delta(G) = 2 \) in Section 5. We obtain more structures of \( \text{sat}(n, C_5) \)-graphs \( G \) with \( \delta(G) = 1 \) in Section 6, and prove that all \( \text{sat}(n, C_5) \)-graphs are presented in Figure 1 or in \( G_1, G_2 \), or \( G_3 \) in Section 7.

**§ 2. Extremal graphs.**

In this section, we present \( \text{sat}(n, C_5) \)-graphs most of which are also constructed by Fisher, Fraughnaugh, and Langley [8]. First, the graph \( H_{7k+1} \), obtained by identifying \( k \) copies of \( G_3^8 \) at \( y \), is essentially the extremal graph for most values of \( n \). Let \( G_{11} \) be the graph obtained from \( G_{12} \) in Figure 1 by removing the degree-1 vertex \( w \). When we add sets of vertices, \( \{u_1, u_2\}, \{v_1, v_2\} \), to obtain graphs in \( G_1 \) and \( G_2 \), their neighbors are \( N(u_1) = \{u_2\}, N(u_2) = \{y_1, y_2, u_1\}, N(v_i) = \{z, v_{3-i}\}, i = 1, 2, \) where \( z \) has \( N(z) = \{y_1, y_2\} \) in \( G_{11} \).

We define \( G_1 = \{G : G \text{ is obtained from } G_{11} \text{ by adding } t \text{ pairs } u_1, u_2 \text{ and } k - t \text{ pairs } v_1, v_2, t \leq k, \text{ and } 0 \leq k \leq 8\} \cup \{G : G \text{ is obtained from } G_{12} \text{ by adding } \ell \text{ pairs } u_1, u_2, 0 \leq \ell \leq 11\} \).
Fig. 1. sat \((n, C_5)\)-graphs

Next we define \(G^2\). Let \(S\) be \(G[A]\) where \(G = G_{18}\) and \(A = \{z_0, \ldots, z_3, w_0, \ldots, w_3\}\) in Figure 1. We obtain \(G_{k+8q+2p}\) from \(G_k\) by adding \(q\) copies of \(S\) and \(p\) copies of \(b_1, b_2\). In \(G_{k+8q+2p}\),
and if \( k = 20 \), then \( p = 0 \). The graph \( G_{k+1} \) is obtained from \( G_{k+1} \in \mathcal{F}_1 \) by first removing \( w^* \), next adding \( r \) pairs \( \beta_1, \beta_2 \), where \( w^* \in N(z^*) \) is the degree-1 \( w \) or \( u_3 \) shown in Figure 1, and \( N(\beta_i) = \{ \beta_3-i, z^* \} \). Let \( \mathcal{F}_2 = \{ G_{k+2r}(w^*) : k + 2r \leq 27 \} \), where \( w^* \) is the degree-1 \( w \) or \( u_3 \) in \( G_{k+1} \in \mathcal{F}_1 \). The graph \( G_{r,k} \) is obtained from \( G_k \in \mathcal{F}_1 \cup \mathcal{F}_2 \) on \( k \) vertices by adding \( r \) pairs of \( u_1, u_2 \) and \( (n-k)/2 - r \) pairs of \( a_1, a_2 \), where \( N(a_2) = \{ y_1, z_0, a_1 \} \) for center \( z_0 \in S \), and \( d(a_1) = 1 \). Let \( \mathcal{F}_3 = \{ G_{r,k}^* : n \leq 34, n-k \) is even, \( r \leq (n-k)/2 \), and \( n \leq 27 \) if \( n \) is odd \}. Finally \( \mathcal{G}_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \).

Now we define \( \mathcal{G}_3 \), the set of the extremal graphs for most values of \( n \). Recall that \( H_{7k+1} \) is obtained by identifying \( k \) copies of \( G_8^2 \) at \( y \). The family \( \mathcal{G}_3 \) consists of all graphs \( G \) obtained from \( H = H_{7k+1} \) by adding \( t \) vertices as described below, where \( k \geq 2 \), \( t \leq 12 \), and if \( t \) is odd, then \( k \geq 3 \) and \( t \leq 5 \). Let \( R = \{ \gamma \in V(H) : d(\gamma) = 4 \} \). We first add \( p \) copies of \( G_7^2 \) and \( q \) copies of \( G_7^2 \) by identifying them and \( H \) at \( y \), where \( p \leq t/6 \) and \( q \leq (t-6p)/6 \). If \( t \) is odd, then we add one degree-2 vertex \( u^* \) with \( N(u^*) = \{ y, \gamma \} \) or \( \{ w, z \} \), where \( \gamma \in R \) and \( N_H(w) = \{ z \} \). Finally we add \( [(t - 6p - 6q)/2] \) pairs of \( b_1, b_2 \), where either \( N(b_i) = \{ b_{3-i}, b \} \) for \( i = 1, 2 \), and \( b \in \{ y \} \cup N(y) \) has no degree-1 neighbor, or \( N(b_2) = \{ y, \gamma, b_1 \} \) and \( d(b_1) = 1 \), where \( \gamma \in R \). If \( t = 5 \), we also obtain \( G \) by identifying either one \( G_6^1 \) or one \( G_6^2 \) and \( H \) at \( y \).

It is left to the reader to verify that each of the graphs in Figure 1 and in \( \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3 \) is \( C_5 \)-saturated.