DECOMPOSITION OF COMPLETE GRAPHS
INTO $k$ FACTORS OF DIAMETER 2

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Abstract. In this paper, we prove that for $k \geq 70$, the complete graph $K_n$ on $n$ vertices can be decomposed into $k$ factors of diameter 2 if and only if $n \geq 6k$.

§ 1. Introduction.

A decomposition of a graph is a collection of subgraphs that partition its edge set. A spanning subgraph or a factor of a graph is a subgraph using all the vertices.

The decomposition problem for complete graphs into factors with given diameters was originally introduced by J. Bosák, A. Rosa and Š. Znám [BRZ]. A graph with diameter 1 is a complete graph. For $d_i \geq 2$, $1 \leq i \leq k$, define $f(d_1, d_2, ..., d_k)$ to be the minimum number of edges such that the complete graph $K_n$ can be decomposed into $k$ factors, $F_1$, $F_2$, ..., $F_k$, with $\text{diam}(F_i) \leq d_i$. This problem can be found in Bollobás’ classic Extremal Graph Theory [Bol]. J. Bosák, A. Rosa and Š. Znám [BRZ] proved that if $m > f(d_1, d_2, ..., d_k)$, then $K_m$ can also be decomposed into factors, $F_1$, $F_2$, ..., $F_k$ with $\text{diam}(F_i) \leq d_i$.

Palumbiny [P] proved that if $d_1, d_2, ..., d_k \geq 3$, then $f(d_1, d_2, ..., d_k) = 2k$. It is easy to see that for a finite number $d \geq 3$, a graph $G$ with diameter at most $d$ is connected, so it contains a spanning tree. Thus such a graph $G$ has at least $n(G) - 1$ edges. Summing the number of edges from the $k$ factors implies that the minimum $n$ such that $K_n$ can be decomposed into $k$ factors of finite diameter is at least $2k$. Palumbiny [P] constructed a decomposition of $K_{2k}$ into $k$ factors of diameter $3$.

Let $f(k)$ denote the minimum $n$ such that $K_n$ can be decomposed into $k$ factors of diameter $2$. For small $k$, Bosák, Rosa and Znám [BRZ] showed $11 < f(3) \leq 13$. Stacho and Urland [SU] proved that $K_{12}$ cannot be decomposed into three factors of diameter $2$. From [BRZ], [Bol], [N], it is known that $17 \leq f(4) \leq 24$, that $22 \leq f(5) \leq 30$ and that $28 \leq f(6) \leq 36$.

For general $k$, Bosák, Erdős and Rosa proved in [BER] that $f(k)$ is finite for all $k \geq 2$, and they obtained also for the first bound, $f(k) \leq 4.9^2k^2\log k$, for all sufficiently large $k$. Sauer [S] proved that $f(k) \leq 7k$. This was improved by Bosák [Bos] to $f(k) \leq 6k$. As for the lower bound,
bound, Bollobás [Bol] showed in 1980 that $f(k) \geq 6k - 9$ for $k \geq 6$. Later Znám improved this to $f(k) \geq 6k - 7$ for $k \geq 664$ [Z1]. Two years later, he proved that $f(k) = 6k$ for $k \geq 10^{17}$ [Z2]. In this paper, we prove that

**Theorem 1.** If $k \geq 70$, then $f(k) = 6k$. That is, for $k \geq 70$, $K_n$ can be decomposed into $k$ factors of diameter 2 if and only if $n \geq 6k$.

We prove Theorem 1 by analyzing the structures of the factors of extremal decompositions. By Bosák's result [Bos], $f(k) \leq 6k$, it is sufficient to show that $K_{6k-1}$ cannot be decomposed into $k$ factors of diameter 2. Write $n = 6k - 1$. Suppose, on the contrary, that such a decomposition exists. Let $F_1, F_2, \ldots, F_k$ be the $k$ factors of a decomposition of $K_n$ into $k$ factors of diameter 2. In section 2, we present some general properties of the $F_i$. In section 3, we present the properties of the factors with many degree-3 vertices. In section 4, we count the number of edges from the $k$ factors and the total number of high-degree vertices to conclude that such a decomposition cannot exist.

**References**


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