DECOMPOSITION OF COMPLETE GRAPHS INTO k FACTORS OF DIAMETER 2

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Abstract. In this paper, we prove that for $k \geq 70$, the complete graph K_n on n vertices can be decomposed into k factors of diameter 2 if and only if $n \geq 6k$.

§ 1. Introduction.

A *decomposition* of a graph is a collection of subgraphs that partition its edge set. A *spanning* subgraph or a *factor* of a graph is a subgraph using all the vertices

The decomposition problem for complete graphs into factors with given diameters was originally introduced by J. Bosák, A. Rosa and Š. Znám [BRZ]. A graph with diameter 1 is a complete graph. For $d_i \geq 2$, $1 \leq i \leq k$, define $f(d_1, d_2, ..., d_k)$ to be the minimum number n, such that the complete graph K_n can be decomposed into k factors, F_1, F_2, ..., F_k with $\text{diam}(F_i) \leq d_i$. This problem can be found in Bollobás' classic *Extremal Graph Theory* [Bol]. J. Bosák, A. Rosa and Š. Znám [BRZ] proved that if $m > f(d_1, d_2, ..., d_k)$, then K_m can also be decomposed into factors, F_1, F_2, ..., F_k with $\text{diam}(F_i) \leq d_i$.

Palumbiny [P] proved that if $d_1, d_2, ... d_k \geq 3$, then $f(d_1, d_2, ..., d_k) = 2k$. It is easy to see that for a finite number $d \geq 3$, a graph G with diameter at most d is connected, so it contains a spanning tree. Thus such a graph G has at least $n(G) - 1$ edges. Summing the number of edges from the k factors implies that the minimum n such that K_n can be decomposed into k factors of finite diameter is at least $2k$. Palumbiny [P] constructed a decomposition of K_{2k} into k factors of diameter 3.

Let $f(k)$ denote the minimum n such that K_n can be decomposed into k factors of diameter 2. For small k, Bosák, Rosa and Znám [BRZ] showed $11 < f(3) \leq 13$. Stacho and Urland [SU] proved that K_{12} cannot be decomposed into three factors of diameter 2. From [BRZ], [Bol], [N], it is known that $17 \leq f(4) \leq 24$, that $22 \leq f(5) \leq 30$ and that $28 \leq f(6) \leq 36$.

For general k, Bosák, Erdős and Rosa proved in [BER] that $f(k)$ is finite for all $k \geq 2$, and they obtained also for the first bound, $f(k) \leq 4.9^2 k^2 \log k$, for all sufficiently large k. Sauer [S] proved that $f(k) \leq 7k$. This was improved by Bosák [Bos] to $f(k) \leq 6k$. As for the lower bound, Bollobás [Bol] showed in 1980 that $f(k) \geq 6k-9$ for $k \geq 6$. Later Znám improved this to $f(k) \geq 6k-7$ for $k \geq 664$ [Z1]. Two years later, he proved that $f(k) = 6k$ for $k \geq 10^{17}$ [Z2]. In this paper, we prove that

Theorem 1. *If $k \geq 70$, then $f(k) = 6k$. That is, for $k \geq 70$, K_n can be decomposed into k factors of diameter 2 if and only if $n \geq 6k$.*

We prove Theorem 1 by analyzing the structures of the factors of extremal decompositions. By Bosák's result [Bos], $f(k) \leq 6k$, it is sufficient to show that K_{6k-1} cannot be decomposed into k factors of diameter 2. Write $n = 6k-1$. Suppose, on the contrary, that such a decomposition exists. Let F_1, F_2, ..., F_k be the k factors of a decomposition of K_n into k factors of diameter 2. In section 2, we present some general properties of the F_i. In section 3, we present the properties of the factors with many degree-3 vertices. In section 4, we count the number of edges from the k factors and the total number of high-degree vertices to conclude that such a decomposition cannot exist.

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