MAT 270 Exam 3
Show work on all questions. If you fail to show work, you will not receive credit.
Good Luck!

1. The position of a particle is given by the function \( s = f(t) = t^3 - 6t^2 + 9t \) where \( t \) is measured in seconds and \( s \) in meters.
   
   a. Find the velocity of the particle after 2 seconds. (6 points)
   
   \[
   v(t) = f'(t) = 3t^2 - 12t + 9 \\
   v(2) = f'(2) = 12 - 24 + 9 = -3 \text{ m/s}
   \]

   b. Find the acceleration of the particle after 4 seconds. (6 points)
   
   \[
   a(t) = f''(t) = 6t - 12 \\
   a(4) = 6(4) - 12 = 24 - 12 = 12 \text{ m/s}^2
   \]

   c. Find the total distance traveled during the first five seconds. (7 points)

   First, find where the particle changes direction by finding where velocity is 0.

   \[
   v = f'(t) = 3t^2 - 12t + 9 = 0 \\
   3(t^2 - 4t + 3) = 0 \\
   3(t - 3)(t - 1) = 0 \\
   t = 3 \text{ or } t = 1
   \]

   Second, evaluate the distance traveled by the particle between times that it turns around.

   \[
   |f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| = 4 + 4 + 20 = 28 \text{ m}
   \]

2. Find the instantaneous rate of change of the area circle when the radius is 3 cm (Show Work). (5 points)

   \[
   A = \pi r^2 \\
   A'(r) = 2\pi r \\
   A'(3) = 6\pi \text{ cm/s}
   \]
3. Given the function, \( f(x) = x^3 + 6x^2 + 9x \), find the following using the tools of calculus. Show Work!
   a. The local maximum and minimum values. (6 points)
   First, find where the derivative is 0. That is where the critical points are.
   \[
   f'(x) = 3x^2 + 12x + 9 = 0
   \]
   \[
   3(x^2 + 4x + 3) = 0
   \]
   \[
   3(x + 3)(x + 1) = 0
   \]
   \( x = -3 \) or \( x = -1 \)
   Second, find the \( y \) values at those points to see if it is a max or min.
   \[
   f(-3) = (-3)^3 + 6(-3)^2 + 9(-3) = 0
   \]
   \[
   f(-1) = (-1)^3 + 6(-1)^2 + 9(-1) = 4
   \]
   LOCAL MAX: (-3,0), LOCAL MIN: (-1,-4)

   b. The absolute minimum and absolute maximum of \( f \) on the interval \([-1, 3]\). (5 points)
   Evaluate for the \( y \)-value at the critical points in the interval and the endpoints.
   \[
   f(-1) = 4
   \]
   \[
   f(3) = 108
   \]
   MAX: (3,108), MIN: (-1,-4)

4. A baseball diamond is a square with 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/sec. At what rate is his distance from second base decreasing when he is halfway to first base? (Draw a Picture—Label you gives and unknowns—Show Work!) (9 points)

- GIVEN: \( \frac{dx}{dt} = 24 \text{ ft/sec} \)
- FIND: \( \frac{dy}{dt} \) when \( x = 45 \)
  \[
  y^2 = (1 - x)^2 + 90^2
  \]
  \[
  \text{when } x = 45, \ y = 45\sqrt{5}
  \]
  \[
  2y \frac{dy}{dt} = 2(1 - x) \left(-\frac{dx}{dt}\right)
  \]
  \[
  \frac{dy}{dt} = \frac{(1 - x) \left(-\frac{dx}{dt}\right)}{y} = \frac{45}{45\sqrt{5}}(-24) = -\frac{24}{\sqrt{5}} \text{ ft/sec}
  \]
5. This morning when coming to work Karen received a ticket from officer Justice on the 101 Loop. The officer claimed that Karen was exceeding the 65 mph speed limit by at least 20 mph. At 10:02:00 she passed officer Justice’s first checkpoint and at 10:08:00 he passed the second checkpoint. According to officer Justice, the distance traveled between checkpoints was 8 miles. Your task is to answer the following questions.

a. Write the formal statement of the Mean Value Theorem and provide an informal statement of what the Mean Value Theorem says? (7 points)

**FORMAL:** If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then there exist a point $c$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**INFORMAL:** It says that if I have a function that is continuous and differentiable between two points, then there is another point somewhere between those two points where the slope is exactly equal to the slope of the secant line between those 2 points.

b. Construct a graph that describes the situation above and explain, using the Mean Value Theorem, what this graph conveys about the speed that Karen was traveling between checkpoints. (Label and place scale marks on your axes). (5 points)

![Graph](attachment:graph.png)

So, the MVT says that at some point between the 2 check points, Karen was driving exactly 80 mph.

c. Using the Mean Value Theorem, what can you say about Karen’s speed over the interval? Is it possible that Karen was not exceeding the speed limit by 20 mph? (3 points)

The mean value theorem proves that Karen was going 80 mph at some point (only 15 over). It is possible that Karen never exceeded the speed limit by more than 20 mph.
6. A 5 ft tall woman walks at a speed of 3 ft/sec away from a 13-ft-tall pole with a street light mounted on top. As the woman walks away, the street light casts a shadow in front of her. How fast is the tip of the shadow moving when she is 30 ft from the pole? How fast is the length of the shadow changing? (Draw a Picture—Label your givens and unknowns—Show Work!) (11 points)

We are given that \( \frac{dx}{dt} = 3 \text{ ft/sec} \). We need to find \( \frac{d(x+y)}{dt} \) when \( x = 30 \).

By similar triangles,

\[
\frac{13}{5} = \frac{x+y}{y} \Rightarrow 13y = 5x + 5y \Rightarrow y = \frac{5}{8}x \Rightarrow \frac{d(x+y)}{dt} = \frac{d(x + \frac{5}{8}x)}{dt} = \frac{13}{8} \cdot 3 = \frac{39}{8} \text{ ft/sec}.
\]

Next, we need to find \( \frac{dy}{dt} \)

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{5}{8} \cdot 3 = \frac{15}{8} \text{ ft/sec}.\]

7. For the function, \( f(x) = x^4 - 4x^3 \) find the:

a. Intervals of increasing; Intervals of decreasing:
(4 points)

\( f'(x) = 4x^3 - 12x^2 = 0 \)
\( x = 0 \text{ or } x = 3 \)

1st DERIVATIVE TEST:

\[
\begin{array}{c|c|c}
& - & + \\
0 & & 3 \\
\end{array}
\]

Decreasing when \( x < 3 \)
Increasing when \( x > 3 \)

b. Concavity and inflection points. (8 points)

\( f''(x) = 12x^2 - 24x = 0 \)
\( x = 0 \text{ or } x = 2 \)

2nd DERIVATIVE TEST:

\[
\begin{array}{c|c|c}
& + & - & + \\
0 & & 2 & \\
\end{array}
\]

Concave up: \((-\infty,0)\) and \((2,\infty)\)
Concave down: \((0,2)\)
c. Draw a rough sketch of the graph of $f$. (4 points)
8. Find an equation of the tangent line to the curve represented by \( y = x + \cos x \) at the point \((0,1)\).

\[ y = x + \cos x \]
\[ y' = 1 - \sin x \]
At \((0,1)\), \(y' = 1 - \sin 0 = 1\)
\[ y - 1 = l(x - 0) \]
\[ y = x + 1 \]

9. Gravel is being dumped from a conveyor belt at a rate of 30 ft\(^3\)/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 7 ft high? (Draw a Picture—Label your givens and unknowns—Show.) (9 points)

\[ \frac{dv}{dt} = 30 \, \text{ft}^3/\text{sec} \]

**Find:** \( \frac{dh}{dt} \) when \( h = 7 \) ft

\[ \frac{h}{2} = r \]
\[ V = \frac{1}{3} \pi r^2 h = \frac{1}{12} \pi h^3 \]
\[ \frac{dV}{dt} = 30 = \frac{1}{4} \pi h^2 \frac{dh}{dt} \]
\[ \frac{dh}{dt} = \frac{30 \cdot 4}{49 \pi} = \frac{120}{49 \pi} \, \text{ft/sec} \]

10. A right circular cylinder is inscribed in a sphere that has a radius of 5 inches. Find the largest possible volume of such a cylinder. (Draw a Picture—Label your givens and unknowns—Show work.) (EXTRA CREDIT—10 points)

TBA