The analytic function \( z \rightarrow \frac{z+1/z}{2} \).

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**Definitions:** Let \( f : \mathbb{C} - \{0\} \rightarrow \mathbb{C} \) be defined by

\[
f(z) = \frac{1}{2} \left( z + \frac{1}{z} \right)
\]

Also let \( D \) be the open unit disk, \( H^+ \) the upper half plane, \( H^- \) the lower half plane and \( D^+ = D \cap H^+, \) \( D^- = D \cap H^- \).

**Result:** \( f \) maps \( D^+ \) 1-1 onto \( H^- \) and \( D^- \) 1-1 onto \( H^+ \).

**Proof:** Let \( z \in \mathbb{C} - \{0\} \) be given by its polar representation \( z = re^{i\phi} \), \( 0 < r, \) \( 0 \leq \phi < 2\pi \). Then

\[
f(z) = \cos \phi \frac{r + \frac{1}{r}}{2} + i \sin \phi \frac{r - \frac{1}{r}}{2}
\]

This shows that \( f(z) \) is real if and only if \( z \) is on the unit circle \(|z| = 1\) or real.

If \( r < 1 \) and \( 0 < \phi < \pi \), then \( r - \frac{1}{r} < 0 \) and \( \sin \phi > 0 \), so the imaginary value of \( f(z) \) is negative. This shows that \( f \) maps \( D^+ \) into \( H^- \). Likewise, one shows that \( f \) maps \( D^- \) into \( H^+ \).

It is now shown that \( f \) is onto. Let \( w \in H^- \) and let \( \sqrt{\cdot} \) be the standard branch of the square root function defined on the ”slit plane” \( \mathbb{C}^* = \mathbb{C} - (-\infty, 0] \):

\[
\sqrt{re^{i\phi}} = \sqrt{r}e^{i\phi/2}
\]

Let

\[
z_1 = w + \sqrt{w^2 - 1}
\]
\[
z_2 = w - \sqrt{w^2 - 1}
\]
These are well-defined since $w^2 - 1 \notin \mathbb{R}$ for $w \in H^-$. It is $z_1z_2 = 1$ and

\[(z - z_1)(z - z_2) = (z - (w + \sqrt{w^2 - 1}))(z - (w - \sqrt{w^2 - 1})) = z^2 - 2w + 1\]

Thus the $z_i$ solve $z^2 + 1 = 2w$ and since they are nonzero, they solve $z + \frac{1}{z} = 2w$. Hence $f(z_i) = w$. Since $w$ is not real, neither $z_i$ can be on the unit circle. Since $z_1z_2 = 1$ and $|z_i| \neq 1$, $z_j \in D$ for exactly one $j \in \{1, 2\}$. But $f$ maps $D^-$ into $H^+$ and $(D \cap \mathbb{R}) - \{0\}$ into $\mathbb{R}$; since $f(z_j) = w \in H^-$, $z_j$ must be in $D^+$. This shows that $f$ maps $D^+$ onto $H^-$.

Since the equation $f(z) = w$ is equivalent to a quadratic equation, each $w \in H^-$ can at most have 2 preimages under $f$. Therefore, $f(z) = w$ only for $z = z_1$ or $z = z_2$. Since the $z_i$ with $i \neq j$ is not in $D$, $f$ as a map from $D^+$ to $H^-$ is 1-1.

Since $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C} - \{0\}$, $f$ maps $D^-$ 1-1 onto $H^+$. 