Evaluating the Moments of a Gaussian Function

Let us evaluate the integrals \( \int_{-\infty}^{\infty} x^k e^{-\frac{x^2}{2}} \, dx \) for integers \( k \), also known as the \emph{moments} of the Gaussian function \( f(x) = e^{-\frac{x^2}{2}} \). We only need to concern ourselves with the even \( k \), since the integral is zero for the odd \( k \)'s by symmetry. Thus we can assume \( k=2n \) for integer \( n \). We split the integrand as follows:

\[
\int x^{2n} e^{-\frac{x^2}{2}} \, dx = \int x^{2n-1} x e^{-\frac{x^2}{2}} \, dx
\]

Observe that \( x e^{-\frac{x^2}{2}} = \frac{d}{dx}(-e^{-\frac{x^2}{2}}) \), so this factor can be the \( v' \) in an integration by parts:

\[
\int x^{2n-1} x e^{-\frac{x^2}{2}} \, dx = x^{2n-1} \cdot \left(-e^{-\frac{x^2}{2}}\right) - \int (2n-1)x^{2n-2} \left(-e^{-\frac{x^2}{2}}\right) \, dx
\]

Simplifying this, we get

\[
\int x^{2n} e^{-\frac{x^2}{2}} \, dx = -x^{2n-1} e^{-\frac{x^2}{2}} + (2n-1) \int x^{2n-2} e^{-\frac{x^2}{2}} \, dx
\]

Now that we got the indefinite integral, let’s apply this to the improper integral over the reals:

\[
\int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2}} \, dx = (2n-1) \int_{-\infty}^{\infty} x^{2n-2} e^{-\frac{x^2}{2}} \, dx
\]

Notice that the term \(-x^{2n-1} e^{-\frac{x^2}{2}}\) has limits of zero as \( x \to \pm \infty \) and therefore contributes nothing.

So we got a recursion formula here: if we define

\[
\int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2}} \, dx = A_n
\]

Then the formula above becomes \( A_n = (2n-1)A_{n-1} \). With the known value \( A_0 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi} \), this recursion determines all \( A_n \). For example:

\[
A_1 = \sqrt{2\pi}
A_2 = 3\sqrt{2\pi}
A_3 = 5 \cdot 3\sqrt{2\pi}
A_4 = 7 \cdot 5 \cdot 3\sqrt{2\pi}
\]