The midpoint derivative property of quadratic functions and its uniqueness

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1 Introduction

A consequence of the mean value theorem of differentiation is that the average (secant) slope of a differentiable function defined on the real line between two points is equal to the tangent slope at some point(s) in between. The theorem is silent on the question of where exactly the points are located, or how many there are.

Quadratic functions are an important special case, because the point guaranteed by the MVT is unique, and always the midpoint. Furthermore, only quadratic functions have that property.

2 Main Results

Theorem 1. Suppose $f(x) = px^2 + qx + r$ is a quadratic function ($p \neq 0$). Then for all distinct real numbers $a, b$

$$
\frac{f(b) - f(a)}{b - a} = f'(c).
$$

(1)

for the unique $c = \frac{a+b}{2}$.

Verify this through direct substitution and algebraic simplification.

It may be surprising that only quadratic functions have this property. Assuming that the second derivative exists, the so-called converse of the theorem just discussed is true.
Theorem 2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and suppose that for all distinct real numbers $x, y$

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

(2)

for $c = \frac{x+y}{2}$, and only for that $c$. Then $f$ must be quadratic.

This is harder to prove. Some ideas: play around with the given equations. Rewrite them, take derivatives of them, combine them. See if you can come up with a condition that says that $f'$ is linear.