The Binomial Theorem
Introduction

You should be familiar with the following formula:

\[(x + y)^2 = x^2 + 2xy + y^2\]

The binomial theorem explains how to get a corresponding expansion when the exponent is an arbitrary natural number.
Deriving the Binomial Theorem

Let us ponder the expression

\[(x + y)^n = (x + y)(x + y) \cdots (x + y)\]

To expand this into a sum, we need to multiply “everything by everything”. More precisely, we need to take a sum of all products of the form

factor chosen from the first parenthesis times
factor chosen from the second parenthesis times
factor chosen from the third parenthesis etc... times
factor chosen from the last parenthesis.

For example, if \(n = 5\), then one of these factors is \(x \cdot x \cdot y \cdot x \cdot y = x^3 y^2\).

Of course, we also get \(x^3 y^2\) by selecting the three \(x\)'s and the two \(y\)'s from different parentheses. Here comes the key idea:

There are \(C(5,3)\) ways of selecting the three parentheses from which we select \(x\). Therefore, there are as many terms \(x^3 y^2\) in the expansion, and therefore, \(C(5,3)\) is the coefficient of the term \(x^3 y^2\) in the expansion.
The Binomial Theorem

Let’s generalize this understanding. In the expansion of $(x + y)^n$, the coefficient of the term $x^k y^{n-k}$ is $C(n, k)$. Therefore, since the expansion contains these (and only these) types of terms for $k = 0$ to $k = n$,

$$(x + y)^n = \sum_{k=0}^{n} C(n, k) x^k y^{n-k}$$

Due to the symmetry of combinations, we can also write this formula as

$$(x + y)^n = \sum_{k=0}^{n} C(n, k) x^{n-k} y^k$$

This form, where the powers of $x$ are decreasing, is more common.

It is also common to use the notation $C(n, k) = \binom{n}{k}$ here. The fact that combinations appear as coefficients in the binomial theorem explains why there are also known as binomial coefficients.
Example 1

\[(x + y)^3 = C(3,0)x^3y^0 + C(3,1)x^2y^1 + C(3,2)x^1y^2 + C(3,3)x^0y^3\]

When we simplify these types of formulas, it is helpful to remember that \(C(n, 0) = C(n, n) = 1\) for all natural numbers \(n\), since there is exactly one way to make an unordered selection of no elements, or all elements, from \(n\) elements.

Furthermore, \(C(n, 1) = C(n, n - 1) = n\) for all natural numbers \(n\), since there are exactly \(n\) ways to make an unordered selection of 1 elements, or all but one elements, from \(n\) elements.

Therefore
Example 2

Expand \((2x - 3y)^4\).

By the binomial theorem, this expression is equal to

\[
(2x - 3y)^4 = \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-3y) + \binom{4}{2}(2x)^2(-3y)^2 + \binom{4}{3}(2x)(-3y)^3 + \binom{4}{4}(-3y)^4
\]

Simplifying this, we get

\[
16x^4 + 4 \cdot 8x^3 \cdot (-3y) + 6 \cdot 4x^29y^2 + 4 \cdot 2x \cdot (-27y^3) + 81y^4
\]

The final simplification of this sum is

\[
(2x - 3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4
\]
Example 3

Simplify

\[
\binom{30}{2} 2^3 + \binom{30}{3} 2^4 + \ldots + \binom{30}{30} 2^{31}
\]

without the use of a calculator.

Observe that this sum has many of the ingredients of a binomial expansion- binomial coefficients and ascending powers of a quantity. We identify \( n = 30 \) and \( y = 2 \). There is no apparent \( x \), which we fix by setting \( x = 1 \). With these settings, the binomial theorem becomes

\[
(1 + 2)^{30} = \sum_{k=0}^{30} \binom{30}{k} 2^k
\]

This is almost equal to the given sum, except for the fact that the “\( k \)” in the powers of 2 is one bigger and therefore does not match the “\( k \)” in the bottom of the binomial coefficients, and the fact that the latter starts at 2, not 0.

The second problem is easily fixed by factoring out a factor of 2:

\[
\binom{30}{2} 2^3 + \binom{30}{3} 2^4 + \ldots + \binom{30}{30} 2^{31} = 2 \left( \binom{30}{2} 2^2 + \binom{30}{3} 2^3 + \ldots + \binom{30}{30} 2^{30} \right) = 2 \sum_{k=2}^{30} \binom{30}{k} 2^k
\]
Example 3, continued

The problem that $k$ in the given sum starts at $k = 2$ is fixed by subtraction:

$$\sum_{k=2}^{30} \binom{30}{k} 2^k = \sum_{k=0}^{30} \binom{30}{k} 2^k - \binom{30}{1} 2^1 - \binom{30}{0} 2^0$$

Simplifying this, we get

$$\sum_{k=2}^{30} \binom{30}{k} 2^k = 3^{30} - 30 \cdot 2 - 1 \cdot 1$$

Therefore, our original sum has been simplified to $2 \cdot (3^{30} - 61)$. 
The Sum of Combinations

Inspired by the previous example, we recognize that interesting special cases of the binomial theorem are obtained for special values of $x$ and $y$.

$x = 1, y = 1$:

$$2^n = \sum_{k=0}^{n} C(n, k)$$

This shows that all the combinations for a given $n$ add up to $2^n$. This confirms our interpretation of $C(n, k)$ as the number of subsets of size $k$ of a set of size $n$. All those numbers must add up to the total number of subsets of a set of size $n$, which we already know to be $2^n$. 
Alternating Sum of Combinations (I)

\[ x = 1, y = -1: \]

\[ 0 = \sum_{k=0}^{n} C(n, k)(-1)^k \]

This shows that the alternating sum of the combinations for a given \( n \) add up to 0.

Since the terms with odd \( k \)'s in this sum have negative signs in front of them, we can move them to the left side and get the following identity:

\[ \sum_{k=0, \text{odd}}^{n} C(n, k) = \sum_{k=0, \text{even}}^{n} C(n, k) \]

This means that for given \( n \), the \( C(n, k) \) with odd \( k \) add up to the same number as the \( C(n, k) \) with even \( k \).

Example: \( n = 5 \)

\[ C(5,1) + C(5,3) + C(5,5) = 5 + 10 + 1 = 16 \]
\[ C(5,0) + C(5,2) + C(5,4) = 1 + 10 + 5 = 16 \]

You may think
Alternating Sum of Combinations (II)

Example: \( n = 5 \)

\[
C(5,1) + C(5,3) + C(5,5) = 5 + 10 + 1 = 16
\]

\[
C(5,0) + C(5,2) + C(5,4) = 1 + 10 + 5 = 16
\]

You may think that this is not a particularly impressive identity after all, since both sums contain the same terms, just in a different order. This is generally true if \( n \) is odd. However, if \( n \) is even, then the terms are different (and there are different numbers of them too), so the fact that their sum is equal is not an obvious fact. Let us take \( n = 6 \) as a second example:

\[
C(6,1) + C(6,3) + C(6,5) = 6 + 20 + 6 = 32
\]
Pascal’s Triangle

There is another interesting relationship between combinations that is usually visualized by arranging the combinations in a triangular fashion called Pascal’s Triangle.

In Pascal’s Triangle, each combination is the sum of the two combinations diagonally above it:

\[ \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \]

for integers \( n \geq 1 \) and \( 1 \leq k < n \).

As an exercise, you should try to prove this using the definition of combinations.