Permutations and Combinations
Introduction

Permutations and combinations refer to number of ways of selecting a number of distinct objects from a set of distinct objects. Permutations are **ordered** selections; combinations are **unordered** selections.

Example: \( S = \{1,2,3\} \).

Ordered selections of two objects from \( S \): 1,2; 1,3; 2,1; 2,3; 3,1; 3,2

Unordered selections of two objects from \( S \): \( \{1,2\} \), \( \{1,3\} \), \( \{2,3\} \)

Observe that there are twice as many permutations as combinations in this case, because each permutation corresponds to two combinations. The general rule for the ratio of permutations and combinations is more complicated. We will discuss it in this presentation.

We write \( P(n,k) \) for the number of permutations of \( k \) objects from \( n \) objects; and \( C(n,k) \) or \( \binom{n}{k} \) (“\( n \) choose \( k \)”) for the number of combinations of \( k \) objects from \( n \) objects. We also refer to the numbers \( C(n,k) \) as **binomial coefficients**. The reason for this will become clear in the presentation on the binomial theorem.
Permutations

Suppose you have \( n \) objects and wish to make an ordered selection of \( k \) objects from them. We refer to these ordered selections as *permutations* and write \( nP_k \) or \( P(n, k) \) for their number.

There are \( n \) choices for the first object. Once that one has been chosen, there are only \( n - 1 \) choices for the second object, then only \( n - 2 \) choices for the third object, and so on. For the \( k \)th object, there are \( n - k + 1 \) choices.

By the product rule, the number of ways of making this ordered selection is

\[
nP_k = P(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1)
\]

A special case is \( n = k \). In this case, we are selecting all available objects, and the number of ways of making that (ordered) selection is

\[
n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1
\]

You may already be familiar with this type of expression. It is called “\( n \) factorial” and written \( n! \).
A formula for permutations

Using the factorial, we can rewrite

\[ P(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1) \]

as

\[ P(n, k) = \frac{n!}{(n - k)!} \]

This formula is theoretically useful, for proving formulas involving permutations (and combinations), but it is of no computational relevance. \( P(n, k) \) is most efficiently computed in a for loop where \( k \) goes from \( n \) down to \( n - k + 1 \), and where all these \( k \) values are multiplied together.

Computing two factorials, only to cancel out most of the factors by division afterwards, is inefficient at best, impossible at worst. The latter is the case when \( n! \) is so large, it goes beyond the available integer data types.
Example 1

Problem: a head of state wishes to receive 5 diplomats. The order of reception is important for diplomatic protocol. In how many different orders can the 5 diplomats be received?

Solution: the question asks for the number of permutations of 5 out of 5 elements. There are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ many such permutations.
Example 2

Problem: a department has 30 professors. Four need to be selected for a committee. The committee has a chair, a vice chair, a secretary, and a fourth member without special privileges or duties. In how many ways can this committee be chosen?

Solution: selecting this committee means making an ordered selection of 4 out of 30 people. Therefore, the number of ways of doing that is

\[ P(30,4) = 30 \cdot 29 \cdot 28 \cdot 27 = 657720 \]
Example 3

Problem: How many 3 letter “words” (that includes nonsense words) can you create from the 26 letters of the alphabet if there is no repetition of letters?

Solution: the number of such letters is $P(26,3) = 26 \cdot 25 \cdot 24 = 15600$. 
Combinations

Suppose you have \( n \) objects and wish to make an unordered selection of \( k \) objects from them. We refer to these selections as combinations and write \( nC_k \) or \( C(n, k) \) for their number.

Let’s not try to re-invent the wheel here. We already know that the number of ways of making an ordered selection is \( P(n, k) \). So now all we need is a relationship between the number of combinations and permutations for \( k \) objects chosen from \( n \) objects. An example will explain this relationship.

Let’s say we have 4 objects: 1,2,3,4, and we are selecting 3 of them.

Then each unordered selection, like \{1, 3, 4\} corresponds to several ordered selections: 1, 3, 4; 1, 4,3; 3,1,4; 3,4,1; 4,1,3; 4,3,1. Since each unordered selection has 3 objects, there are 3! ordered selections (permutations) of those 3 objects.

This means that the number of ordered selections (permutations) is equal to \( k! \) times the number of unordered selections (combinations):

\[
P(n, k) = C(n, k)k!
\]
Combinations, Continued

The formula

\[ P(n, k) = C(n, k)k! \]

can be solved for the number of combinations \( C(n, k) \):

\[ C(n, k) = \frac{P(n, k)}{k!} = \frac{n!}{(n - k)! k!} \]

Again, the formula involving factorials is of purely theoretical value. It is not efficient for computing combinations in practice. In practice, we compute combinations by using the middle formula. We compute the corresponding number of permutations and then divide by \( k! \).

Combinations usually involve a large number of cancellations that can be exploited for computing them without a calculator. Example:

\[ C(5,3) = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 5 \cdot 2 = 10 \]
Example 1

How many 3 card hands can you be dealt from a standard deck of 52 playing cards?

A hand of cards is by definition an unordered selection. Therefore, the answer to the question is

$$C(52,3) = \frac{52 \cdot 51 \cdot 50}{3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 50 = 22100$$
Example 2

How many committees of 4 senators can be selected from 100 senators?

Without additional requirements, these committees correspond to unordered selections. Therefore, the answer is

$$C(100,4) = \frac{100 \cdot 99 \cdot 98 \cdot 97}{4 \cdot 3 \cdot 2 \cdot 1} = 25 \cdot 33 \cdot 49 \cdot 97 = 3,921,225$$

If there was the additional requirement that a committee has a chair, then the answer would be, according to the product rule,

$$100 \cdot C(99,3) = 15,684,900$$
Example 3

Problem: There are 193 diplomats at a meeting, each representing one nation. In how many ways can a committee of 20 diplomats be selected so that exactly 4 of them are from the 55 African countries? Give the answer purely in terms of combinations.

Answer: $C(138, 16) \cdot C(55, 4)$
The symmetry of combinations

Combinations have an important symmetry. Observe that when you replace \( k \) by \( n - k \) in the formula

\[
C(n, k) = \frac{n!}{(n - k)! \cdot k!},
\]

the two factors in the denominator just switch places. Therefore, \( C(n, k) = C(n, n - k) \).

This formula leads to a computational advantage. When we compute a combination, we need to evaluate a product of \( k \) factors in the numerator, and \( k \) factors in the denominator. Therefore, the computational expense grows with \( k \). Using \( C(n, k) = C(n, n - k) \), we can effectively replace a \( k \) which is bigger than one-half of \( n \) by one that is smaller than that.

Example: \( C(10,8) = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = C(10,2) = \frac{10 \cdot 9}{2 \cdot 1} = 5 \cdot 9 = 45. \)
Another perspective on combinations

An unordered selection of objects from a set $S$ is a subset of $S$. Therefore, $C(n, k)$ is the number of subsets of size $k$ of a set of size $n$.

Example: How many subsets containing 3 members are there of a set of 6 members?

Answers: There are $C(6,3) = \frac{6\cdot5\cdot4}{3\cdot2\cdot1} = 5 \cdot 4 = 20$ such subsets.

This perspective leads to an effortless explanation of the symmetry of combinations. Picking a subset of size $k$ of a set of size $n$ is equivalent to picking its complement, which is of size $n - k$. Deciding what is included in your set is equivalent to deciding what is excluded, so the number of ways to pick the excluded set must be equal to the number of ways to pick the included set. Therefore, $C(n, n - k)$ must be equal to $C(n, k)$. 
Distinguishable Permutations

An important application of combinations is to compute numbers of *distinguishable permutations*.

We speak of distinguishable permutations when we consider rearrangements of objects where identical copies are present. The following example illustrates that.

How many distinguishable rearrangements are there of the string TTTFF?

The string contains 3 identical T’s, and 2 identical F’s.

A unique, distinguishable rearrangement of the string is completely determined by the positions of the 3 T’s, which is a set of 3 numbers. For example, TTTFF above corresponds to the position set \{1,2,3\}. TTFTF corresponds to the position set \{1,2,4\}.

Therefore, the number of distinguishable rearrangements of TTTFF is equal to the number of ways we can pick a subset of 3 numbers out of the set \{1,2,3,4,5\}. By what we just learned, that can be done in exactly \(C(5,3)\) ways.

[We could also define each distinguishable rearrangement by the set of positions of the F’s. By the same logic we just used, the number of distinguishable rearrangements of TTTFF is then \(C(5,2)\). Due to the symmetry of combinations, \(C(5,3)\) and \(C(5,2)\) are equal.]