Sets
Definitions and Vocabulary of Sets

A **set** is an unordered collection of distinguishable objects. These objects are called **members** or **elements** of the set. Sets are usually denoted by uppercase letters, and their members by lowercase letters.

To say that \( a \) is a member of the set \( A \), we write \( a \in A \). The negation of that statement is written as \( a \notin A \).

To define a finite set (a set with finitely many elements), we often just list the members between braces:

\[
A = \{1, 2, 3\}
\]

This is the set with the numbers 1, 2 and 3 in it.

The members of a set don’t have to be numbers, and don’t have to be “of the same type”. They can even be sets themselves. The following is a valid set:

\[
A = \{-1.3, \Delta, \text{my dog, Tuesday, \{1,2\}}\}
\]
A set is an unordered data structure

It bears repeating that a set is fundamentally an unordered data structure. There is inherently no “first” element of the set, no “last” element, and no “next” element for any given element.

\[ \{\Delta, *\} = \{*, \Delta\} \]

It is true that certain sets, such as the real numbers, contain an additional mathematical structure called an order. But this is not inherent in the concept of a set.
The members of a set are distinguishable objects

Imagine you are planning a party, and are going to invite 5 people. One of these people is your friend Jane. By accident, you wrote Jane’s name twice on your list. This list that describes the set $S$ of people to invite may look like this:

$$S = \{\text{Jane, Peter, Mark, Jane, Eric, Lisa}\}$$

There are only 5 people in this set. A person can only be a member of the set, or not be a member. Having Jane listed a second time as a member is redundant and just reaffirms her membership. Therefore,

$$S = \{\text{Jane, Peter, Mark, Eric, Lisa}\}$$

The general lesson here is that an object cannot be a member of a set more than once. This leads to set equations that may at first seem counter-intuitive:

$$\{\Delta, \Delta, \Delta, \Delta\} = \{\Delta\}$$
Other Ways of Defining Sets

When a set has more than a few members, it can be convenient to define it with “…” notation:

\[ A = \{1, 2, 3, 4, \ldots, 100\} \]

\( A \) is the set of consecutive integers from 1 to 100. This notation can be problematic when the intended pattern is unclear. For example, does \( B = \{1, 2, \ldots, 128\} \) mean that \( B \) is the set of consecutive integers from 1 to 128, or is it perhaps the set of powers of two in this range?

To avoid ambiguity, we can use set builder notation, in which we state a condition that characterizes membership in the set:

\[ S = \{x \mid \text{the condition on } x \text{ that defines the membership}\} \]

For example, the set \( A \) above could have been defined with set builder notation as

\[ A = \{x \mid x \text{ is an integer and } 1 \leq x \leq 100\}. \]

Make sure not to write equations like \( A = 1 \leq x \leq 100 \) or \( A = (1 \leq x \leq 100) \). \( A \) is a set, and a set cannot be equal to a number, or to a statement. You must carefully distinguish between a set and a statement describing the membership condition of the set.
Do not Abuse Set Notations and Language

If you know that Pierre is French, you say “Pierre is French”. Unless there is a specific need, you do not say “Pierre is a member of the set of all people who are French”.

Likewise, you should not compulsively communicate mathematical information in the language of sets. If you know that an object $x$ satisfies a property $P(x)$, do not make that statement in the form of “$x$ is a member of the set of all objects that satisfy $P$”. Simply write $P(x)$.

For example, instead of writing $x \in \{x|x > 1\}$, write $x > 1$.

Avoid tautological constructs of the type $A = \{x|x \in A\}$. It is true that every set is equal to the set of all of its members, but that perspective adds notational complexity without providing specific information about the set $A$. For example, the set $\{x|x \in [1,2]\}$ is more clearly written simply as $[1,2]$. 
The following set names are universally agreed upon on mathematics:

\[ \mathbb{N} = \{1, 2, 3, \ldots \} \text{ is the set of natural numbers.} \]
\[ \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots \} \text{ is the set of integers.} \]
\[ \mathbb{Q} \text{ is the set of rational numbers.} \]
\[ \mathbb{R} \text{ is the set of real numbers.} \]
\[ \mathbb{C} \text{ is the set of complex numbers.} \]

The definition of the natural numbers is contested. Some people and textbooks include the number zero in the definition. We will use the notation \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \} \).
The empty set, singleton sets

\[ \emptyset = \{ \} \] is the empty set, which is the set that contains no members.

A set that has exactly one member is called a singleton set. Example of singleton sets:

\[
A = \{ 1 \} \\
B = \{ \{ 1, 2 \} \} \\
C = \{ \emptyset \}
\]
Rational vs Real Numbers

We already defined the rational numbers in an earlier presentation. They are the numbers that can be written as quotients of integers.

One of the great achievements of ancient Greek mathematics was the proof that there are quantities that are irrational (not rational), such as $\sqrt{2}$. The constants $e$ and $\pi$ are also irrational.

The set of real numbers is the set of rational and irrational numbers. A precise axiomatic definition of the real numbers is beyond the scope of this class.

We can characterize rational and irrational numbers by their decimal expansion. Every real number has a (potentially infinite) decimal expansion. For a rational number, this decimal expansion terminates or is periodic. For an irrational number, the decimal expansion is non-terminating and non-periodic. For example,

$$\pi = 3.14159265358979323846264338327950288419716939937510...$$

It was proved in modern times that in a certain sense, “almost all” real numbers are irrational. You can guess why this might be true: picking a “random” real number means an infinite process of picking each digit randomly from left to right. The probability that this will produce all zeroes after a certain point, or an eternal repetition, seems very small. In fact, it is exactly zero.
Interval notation

A set of all real numbers between two fixed numbers that includes or excludes each of these two numbers is called an interval. We distinguish open and closed intervals. An open interval is an interval that does not contain its own end points:

\[(a, b) = \{x \in \mathbb{R} | a < x < b\}\]

A closed interval is an interval that contains its own end points:

\[[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}\]

There are also intervals which are half-open, half-closed:

\[(a, b] = \{x \in \mathbb{R} | a < x \leq b\}\]

\[[a, b) = \{x \in \mathbb{R} | a \leq x < b\}\]
Common misunderstandings related to intervals

Some students confuse the set \([a,b]\) with the set \(\{a,b\}\). The first is the set of all real numbers from \(a\) to \(b\), including the endpoints. That is an infinite set. The second is the set just containing the two numbers \(a\) and \(b\).

A second and perhaps psychologically related misunderstanding is to think that intervals are sets of integers. According to this misunderstanding, the interval \((1,4)\) is the same as \([2,3]\). It is not. The first interval contains infinitely many real numbers that the second does not contain, namely the ones strictly between 1 and 2, and strictly between 3 and 4.
Again: Do Not Confuse Sets with Statements that Define Them

A common beginner’s mistake is to confuse sets with statements that define the sets and write incorrect statements like this:

\[(2,3) = 2 < x < 3.\]

This is false because it implies that \((2,3) = 2\). That equation asserts that a set is equal to a number. Numbers and sets are different categories of objects. A number can never be equal to a set.

Writing \((2,3) = (2 < x < 3)\) just replaces the problem by a new one: now it is being asserted that a set and a statement are equal, which is just as nonsensical as saying that a set and a number are equal.
Integers and Inequalities

The issue of how inequalities interact with integers is related to the second misunderstanding we discussed on the previous page.

For integers, and for integers only, strict inequalities can be rewritten as non-strict inequalities and vice versa. For example, if $n$ is an integer, then $n > 2$ if and only if $n \geq 3$.

Generally speaking, if $n$ and $m$ are integers, then $n > m$ if and only if $n \geq m$, and $n < m$ if and only if $n \leq m$.

It cannot be emphasized enough that this rule does not hold for general real numbers $n$ and $m$. 
The Subset Relationship

If every element of the set $A$ is also in $B$, we say that $A$ is a subset of $B$ and write $A \subseteq B$.

The formal definition is:

$$A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$$

If $A \subseteq B$ and there is an element in $B$ that is not in $A$, we say that $A$ is a proper subset of $B$ and write $A \subset B$. Examples:

$\{1,2\} \subseteq \{1,2,3\}$ and $\{1,2\} \subset \{1,2,3\}$.

Every set is a subset of itself, but no set is a proper subset of itself. The empty set is a proper subset of every set except itself.
Set Equality

If $A \subseteq B$ and $B \subseteq A$, we say that $A$ and $B$ are equal and write $A = B$. We can formally define set equality by

$$(A = B) \iff \forall x (x \in A \iff x \in B)$$

or by

$$(A = B) \iff (A \subseteq B) \land (B \subseteq A)$$

To prove that two sets $A$ and $B$ are equal, we must prove $\forall x (x \in A \iff x \in B)$. Sometimes, this can be shown through a string of logical equivalences. If we cannot find such a string of equivalences, it may be possible to show separately that the assumption $x \in A$ leads to $x \in B$, and that the assumption $x \in B$ leads to $x \in A$. 
Venn Diagrams for subset relations

It is customary to represent relationships between sets by Venn Diagrams: (usually) circles represent sets, and their overlap represent what they have in common. On the right is a Venn diagram for the subset relation $A \subseteq B$. 
Cardinality

If $S$ is a finite set, the number of elements of $S$ is called the **cardinality** of $S$ and written $|S|$. Examples:

The cardinality of the empty set is zero.

The cardinality of any singleton set is one.

$|\{1,4,6\}| = 3$.

There is a concept of cardinality of an infinite set as well, which we will cover in the presentation on functions.

The notation $c(S)$ is also in use for $|S|$.
Power Sets

The power set of a set $S$, written $\mathcal{P}(S)$ is the set of all subsets of $S$:

$$\mathcal{P}(S) = \{A|A \subseteq S\}$$

Observe that the power set is a set of sets. Examples:

$$\mathcal{P}({1}) = \emptyset, \{1\}$$
$$\mathcal{P}({1,2}) = \emptyset, \{1\}, \{2\}, \{1,2\}.$$ 

Things get interesting and complicated when we take power sets of power sets. For example,

$$\mathcal{P}(\mathcal{P}({1})) = \mathcal{P}(\emptyset, \{1\}) = \emptyset, \emptyset, \emptyset, \{\emptyset, \{1\}\}$$

It is critical to understand here that the set $\{\emptyset\}$ is not the empty set. It is a set with one element, and that one element is the empty set. If you think of sets as boxes, and the empty set as an empty box, then $\{\emptyset\}$ is a box with an empty box in it.

We will later prove that $|\mathcal{P}(S)| = 2^{|S|}$ for all finite sets $S$. 
Working with Sets of Sets of Sets

On the previous page, we had an example of a set that contained sets that contained sets. That can get confusing. In that case, work with appropriate variables. For example, to find

$$\mathcal{P}(\{\emptyset, \{1\}\})$$

we may introduce $$x = \emptyset$$ and $$y = \{1\}$$ for the two members of the set. This lets us focus on $$x$$ and $$y$$ as objects, and forget temporarily that they are also sets. Now it is easier to find the indicated power set:

$$\mathcal{P}(\{\emptyset, \{1\}\}) = \mathcal{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$$

The last step is symbolic back-substitution:

$$\mathcal{P}(\{\emptyset, \{1\}\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\} = \left\{\emptyset, \emptyset, \{\{1\}\}, \emptyset, \{1\}\right\}$$
The Cartesian Product

An **ordered pair** is an ordered list of two objects: \((a, b)\). We can think of an ordered pair of two numbers as a point. (Here, the notation \((a, b)\) does **not** refer to an open interval.) The set of all ordered pairs \((a, b)\), where \(a\) is in some set \(A\), and \(b\) is in some set \(B\) is called the Cartesian product of \(A\) and \(B\):

\[
A \times B = \{(a, b) | a \in A \land b \in B\}
\]

For example, if \(A = \{1,2\}\) and \(B = \{3,4\}\) then

\[
A \times B = \{(1,3), (1,4), (2,3), (2,4)\}
\]

Notation detail matters here. \(\{\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}\) is a different set than \(A \times B\).
Visualization of the Cartesian Product

We can think of $A \times B$ as the set of all points with x coordinate in $A$ and y coordinate in $B$. This set of point will always be a rectangular grid of points.

For example, if $A = \{0,1,2\}$ and $B = \{1,2\}$ then $A \times B$ can be visualized as the grid of red points in the picture on the right.
The Cartesian product of more than two sets

An **ordered n-tuple** is an ordered list of $n$ objects: $(a_1, a_2, \ldots, a_n)$. We can think of an ordered n-tuple as a point in $n$-dimensional space.

The set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_k$ is in some set $A_k$ for all $k = 1..n$ is called the Cartesian product of these sets: $A_1 \times A_2 \times \cdots \times A_n$.

The notations $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ are common.
The Cartesian Product is not commutative!

If $A = \{1,2\}$ and $B = \{3,4\}$ then

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$B \times A = \{(3,1), (3,2), (4,1), (4,2)\}$

Geometrically, we can think of $B \times A$ as the reflection of $A \times B$ across the main diagonal (which switches the role of x and y coordinates.)