Advanced partial fraction example

Finding the partial fractions of a rational function requires a factorization of the denominator. This can be a challenge, as the following example shows:

\[ f(x) = \frac{1}{x^4 + 1} \]

The denominator has no real zeros, so by a version of the fundamental theorem of calculus, the denominator can be factored into two irreducible quadratics:

\[ x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d) \]

1. Explain why we can assume from the outset that the leading coefficients are 1. Why don’t we risk missing the solution by making this assumption?

2. Use the method of comparing coefficients to find the unknown coefficients \( a, b, c, d \).

One you have the factorization of \( x^4 + 1 \), you can find the partial fractions. That, unfortunately, seems to require solving a \( 4 \times 4 \) system of equations, which is computationally expensive. There is a way to reduce the calculation to 2 unknowns.

3. Write down the form of the partial fraction expansion of \( \frac{1}{x^4+1} \) and then replace \( x \) by \( -x \) in that equation. Notice the symmetry, and exploit it to eliminate two unknowns. Then solve the \( 2 \times 2 \) system.

4. Split each partial fraction into a part that can be integrated using the natural log, and one that can be integrated using arctan. Then integrate them. Hint for the arctan parts: complete the square.