Common algebra mistakes

It could be said that most common algebra mistakes are variations of a simple misunderstanding, namely the idea that all functions are linear.

Linearity of a function means that you can break up the function applied to a sum into the sum of the functions applied to the individual parts, and that you can pull out constant factors:

\[ f(x + y) = f(x) + f(y) \quad \text{and} \quad f(cx) = cf(x) \]

for all \( x, y \) and constants \( c \).

Linearity is a nice and powerful property, but most functions don’t have it. Geometrically speaking, only functions whose graphs are straight lines through the origin are linear. This means that power functions (with power not equal to 1), trigonometric functions, inverse trig functions and log functions are NOT linear and breaking up sums as indicated, or pulling out constants, is almost always wrong.

Student Exercises:

1. Find numbers \( a, b \) to demonstrate that, generally, \( \frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y} \), in other words, that the reciprocal function is not linear.

2. What happens to a fraction when you make the denominator bigger? And what happens when you make the denominator smaller? Based on that, give an argument why \( \frac{1}{x+y} = \frac{1}{x} + \frac{1}{y} \) can have NO positive solutions at all.

3. (Harder) Simplify the mistaken identity \( \frac{1}{x+y} = \frac{1}{x} + \frac{1}{y} \) to show that this equation cannot hold for any real numbers. Hint: complete a square.
4. Find numbers \( a, b \) to demonstrate that, generally, \( \sqrt{a + b} \neq \sqrt{a} + \sqrt{b} \). For a particularly nice demonstration, try to find square numbers \( a, b \) so that their sum is also a square!

5. Simplify the mistaken identity \( \sqrt{a + b} = \sqrt{a} + \sqrt{b} \) to discover for which numbers \( a, b \) it is true (there are some!) Is that a contradiction?

6. Recall the range of the sine function and make an argument, based on that, why \( \sin(x + y) \) generally can’t possibly always be equal to \( \sin(x) + \sin(y) \). Be specific and give example values for \( x \) and \( y \).

7. What is the correct formula for \( \sin(x + y) \)? For \( \cos(x + y) \)?

8. If the functions \( f(x) = \sqrt{x + y} \) and \( g(x) = \sqrt{x} + \sqrt{y} \) were equal for all \( x \geq 0 \), for all positive constants \( y \), then their derivatives would have to be equal too! Show that that is not the case. Apply the same idea to the reciprocal function \( f(x) = \frac{1}{x} \), to the sine function and to the natural logarithm function.

9. Consider and address the counter-argument that \( \sin(x + y) \) must be equal to \( \sin(x) + \sin(y) \) for all \( x \) and \( y \) due to the distributive law.