Calculus I Practice Exam 1A

This practice exam emphasizes conceptual connections and understanding to a greater degree than the exams that are usually administered in introductory single-variable calculus courses. It is designed to guide students who are taking such courses to a deeper mastery of the material.

While a number of questions here are fairly typical for actual examinations, you should not infer from the expression “practice exam” that exams encountered in introductory single-variable calculus courses will ask the same types of questions.

Multiple choice:

1. Evaluate the limit algebraically: \( \lim_{x \to 49} \frac{2\sqrt{x} - 14}{49 - x} \)
   
   A) 0       B) \( \frac{1}{7} \)       C) \( -\frac{1}{7} \)       D) \(-\infty\)       E) None of these choices.

2. Evaluate the limit algebraically: \( \lim_{x \to 3} \frac{3x - 9}{\sqrt{7(x^2 - 9)}} \)
   
   A) 0       B) \( \frac{1}{2\sqrt{7}} \)       C) \( \frac{1}{\sqrt{7}} \)       D) \( \frac{17}{90} \)       E) None of these choices.

3. Evaluate the limit algebraically: \( \lim_{x \to 47} \frac{\sqrt{x + 2} - 7}{x - 47} \). DNE means “does not exist”.
   
   A) \( \frac{1}{14} \)       B) \( \frac{\sqrt{47}}{96} \)       C) \( \frac{7}{43} \)       D) DNE       E) None of these choices.
4. Consider the following graph of a function $f$. Then which one of the following statements describes the behavior of the function near $x = a$?

A) $\lim_{x \to a^-} f(x)$ exists, $\lim_{x \to a^+} f(x)$ exists, $\lim_{x \to a} f(x)$ does not exist, $f(a)$ exists.

B) $\lim_{x \to a^-} f(x)$ exists, $\lim_{x \to a^+} f(x)$ does not exist, $\lim_{x \to a} f(x)$ does not exist, $f(a)$ exists.

C) $\lim_{x \to a^-} f(x)$ exists, $\lim_{x \to a^+} f(x)$ exists, $\lim_{x \to a} f(x)$ does not exist, $f(a)$ does not exist.

D) $\lim_{x \to a^-} f(x)$ exists, $\lim_{x \to a^+} f(x)$ does not exist, $\lim_{x \to a} f(x)$ does not exist, $f(a)$ does not exist.

E) None of the others.

5. For which value of the constant $c$ is the function $f$ continuous?

$$f(x) = \begin{cases} cx^2 + 1 & \text{for } x \leq 1 \\ x - c & \text{for } x > 1 \end{cases}$$

A) $c=-1$    B) $c=0$    C) $c=1$    D) $c=2$    E) None of the others
6. Explain why the following function is discontinuous at x=a:

A) Both limit and function value exist, but are not equal to each other.
B) Neither limit nor function value exist.
C) Even though the limit exists, the function value does not.
D) Even though the function value exists, the limit does not.
E) Both limit and function value exist and are equal to each other.

7. Evaluate the following limit in terms of the constant c: \( \lim_{x \to \infty} \left( \sqrt{9x^2 + cx} - 3x \right) \)

A) 0  B) \( \frac{c}{9} \)  C) \( \frac{c}{3} \)  D) \( \frac{c}{6} \)  E) \( \frac{c}{2} \)

8. Evaluate the following limit: \( \lim_{x \to -\infty} \frac{\sqrt{2x^{10} + x}}{2x^5 + 1} \)

A) -1  B) 1  C) \( -\frac{1}{\sqrt{2}} \)  D) \( \frac{1}{\sqrt{2}} \)  E) None of the others
9. Suppose \( f(x) = -2g(x) + \sin x - \cos x + \frac{1}{2\sqrt{x}} \) and \( g'(\pi) = 1 \). Evaluate \( f'(\pi) \).

A) \( f'(\pi) = -1 - \frac{1}{4\sqrt{\pi^3}} \)

B) \( f'(\pi) = -3 - \frac{1}{4\sqrt{\pi^3}} \)

C) \( f'(\pi) = -1 + \sqrt{\pi} \)

D) \( f'(\pi) = -3 + \sqrt{\pi} \)

E) None of the others

10. The following table shows values of the position function of an object that is moving on a straight line.

<table>
<thead>
<tr>
<th>Time in seconds</th>
<th>Position in meters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.5</td>
</tr>
<tr>
<td>2</td>
<td>4.7</td>
</tr>
<tr>
<td>4</td>
<td>7.9</td>
</tr>
<tr>
<td>7</td>
<td>8.5</td>
</tr>
<tr>
<td>9</td>
<td>10.2</td>
</tr>
</tbody>
</table>

What was the average velocity of this object from \( t=1 \) to \( t=9 \) seconds in meters per second?

A) 0.7444

B) 0.9625

C) 0.8375

D) 4.35

E) None of the above.
Free response

1. At what point does the normal to $y = x^2 - x + 1$ at $(1,1)$ intersect the parabola a second time?

2. Based on the graph of $f$ given below, determine at which values of $x$ the quantity 
$$ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} $$
does not exist, and explain why for each value.

3. Let $f(x) = \frac{1}{x+2}$. Use the limit definition of the derivative to find $f'(x)$. 
Answers:

Multiple Choice:

1C  2B  3A  4A  5B  6A  7D  8C  9B  10C

Free Response:

1. To find the normal, we take the derivative of \( y = x^2 - x + 1 \):

\[
y' = 2x - 1
\]

and evaluate that at \( x=1 \): \( y' = 1 \). Therefore, the slope of the tangent of the function at \( (1,1) \) is 1, which means that the slope of the normal (the line perpendicular to the tangent) is -1.

To find the equation of the normal, we use the fact that the normal also intersects \( (1,1) \) and plug that point into the point-slope form:

\[
y - y_1 = m(x - x_1)
\]

\[
y - 1 = -1(x - 1)
\]

Simplifying that, we get \( y = 2 - x \) as the equation of the normal. This line intersects the given parabola at \( (1,1) \) by design, and supposedly intersects it a second time. To find this intersection point, we set the normal and the parabola equal to each other:

\[
x^2 - x + 1 = 2 - x
\]

Simplifying this, we get \( x^2 = 1 \) which means \( x = \pm 1 \).

The solution \( x = 1 \) corresponds to the intersection point \( (1,1) \) that we already know. Therefore, \( x = -1 \) is the location of the intersection we are seeking.

We find the \( y \) coordinate of the intersection point by plugging \( x = -1 \) into \( y = 2 - x \) and find \( y = 3 \). For verification purposes, we may confirm that we get the same \( y \) value by plugging the \( x \) value into \( x^2 - x + 1 \).
Therefore, the second intersection point of the normal with the quadratic is (-1,3).

2. The quantity \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \) is the derivative of \( f \) at \( x \). This does not exist at

\( x=1 \) because \( f \) has a corner there (different limit of the difference quotient from the left and the right.)

\( x=2 \) because \( f \) has a jump discontinuity there (the derivative of \( f \) does not exist where \( f \) is discontinuous.)

\( x=3 \) because \( f \) has a vertical tangent there, so the limit of the difference quotient is infinite, i.e. does not exist.

3. We use the limit definition of derivative:

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left( \frac{x}{x+2} - \frac{1}{x+2} \right)
\]

We simplify the double fraction by creating a common denominator in the numerator and combining the fractions there. Then we cancel the common factor \( h \), which removes the removable discontinuity at \( h = 0 \) and evaluate the limit by substituting \( h = 0 \):

\[
 = \lim_{h \to 0} \frac{x + h + 2}{h} \cdot \frac{1}{x+2} - \lim_{h \to 0} \frac{x + h + 2}{h} \cdot \frac{1}{x+2} =
\]

\[
 = \lim_{h \to 0} \frac{-h}{(x + h + 2)(x+2)} = \lim_{h \to 0} \frac{-1}{(x + h + 2)(x+2)} = -\frac{1}{(x+2)^2}
\]