PRACTICING CONVERGENCE ARGUMENTS

EXERCISES

1. Prove that the sequence \((x_n)\), defined by \(x_1 = \sqrt{2}\) and \(x_{n+1} = \sqrt{2 + x_n}\), converges, and find its limit.

2. Prove, using the definition of convergence, that if \((a_n)\) is a bounded sequence and \((b_n)\) converges to zero, then the sequence \((a_n \cdot b_n)\) converges also to zero.

3. Give an example of sequences \((a_n)\) bounded and \((b_n)\) convergent such that \((a_n \cdot b_n)\) does not converge.

4. Let \(a_0\) and \(b_0\) be real numbers, \(0 < a_0 < b_0\). Define two sequences as follows:
   \[a_{n+1} = \sqrt{a_n b_n}\]
   \[b_{n+1} = \frac{a_n + b_n}{2}\]
   Prove:
   a. \(0 < a_n \leq b_n\) for all natural numbers.
   b. \((a_n)\) is increasing; \((b_n)\) is decreasing.
   c. \(a_n \leq b_k\) for any indices \(n\) and \(k\).
   d. Both sequences are convergent and have the same limit.
   This common limit is called the arithmetic-geometric mean of \(a_0\) and \(b_0\).