Homework problems are due in writing on Wednesday, June 12th.

IN-CLASS EXERCISES

1. For the following, let \( f: X \rightarrow Y \) be an arbitrary function. Decide whether the following statements are true for all sets \( A, B \subseteq X, C, D \subseteq Y \).

   If the statement is true, prove it. If the statement is false, give a counterexample, then find an additional condition that makes it true, and prove it.

   a. \( f^{-1}(f(A)) \subseteq A \).
   b. \( f^{-1}(f(A)) \supseteq A \).
   c. \( f(f^{-1}(C)) \subseteq C \).
   d. \( f(f^{-1}(C)) \supseteq C \).
   e. \( f(A) \cap f(B) \subseteq f(A \cap B) \).
   f. \( f(A) \cap f(B) \supseteq f(A \cap B) \).
   g. \( f(A) \cup f(B) \subseteq f(A \cup B) \).
   h. \( f(A) \cup f(B) \supseteq f(A \cup B) \).
   i. \( f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \).
   j. \( f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \).

2. Prove the following Law of De Morgan:

   If \( (A_i)_{i \in I} \subseteq X \) is a family of sets (\( I \) is any index set), then \( (\bigcap_{i \in I} A_i)^C = \bigcup_{i \in I} A_i^C \).

HOMEWORK PROBLEMS

3. Find the negation of the following quantified statements. Translate both the original and the negated statement into English. Which statements are true, and which are false?

   a. \( \forall x \in \mathbb{R} \exists n \in \mathbb{N} \) such that \( \forall m \in \mathbb{N} \) with \( m \leq x \): \( m \leq n \).
   b. \( \forall x \in \mathbb{R} \exists y \in \mathbb{R} \) such that \( xy < 0 \).
   c. \( \forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists z \in \mathbb{R} \) such that \( x < z < y \).
   d. \( \forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists z \in \mathbb{R} \) such that \( x \leq z \leq y \).
   e. \( \forall a \in \mathbb{R} \forall b \in \mathbb{R} \exists M \in [0, \infty) \) such that \( \forall x \in [a, b] \): \( x^2 \leq M \).

4. Use quantifiers to describe the condition that a set \( G \subseteq X \times Y \) must satisfy to be the graph of a function \( f: X \rightarrow Y \).
5. Let \( f: X \to Y \) be a function. Recall that composition of functions is defined as follows: \((f \circ g)(x) = f(g(x))\).

Prove:
   a. \( f \) is injective if and only if there is a function \( g: Y \to X \) such that \( g \circ f = id_X \).
   b. \( f \) is surjective if and only if there is a function \( g: Y \to X \) such that \( f \circ g = id_Y \).

6. Let \( f: X \to Y \) and \( g: Y \to Z \) be functions. Prove or disprove:
   a. If \( f \circ g \) is injective, then \( g \) must be injective.
   b. If \( f \circ g \) is injective, then \( f \) must be injective.
   c. If \( f \circ g \) is bijective, then \( g \) must be injective and \( f \) must be surjective.
   d. If \( g \) is surjective, and \( f \) is bijective, then \( f \circ g \) is bijective.