PROMOTING AND CHARACTERIZING THE PROBLEM SOLVING
BEHAVIORS OF PROSPECTIVE HIGH SCHOOL MATHEMATICS TEACHERS

by

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ABSTRACT

This dissertation reports on one effort to improve the mathematical preparation of prospective secondary mathematics teachers (PSMTs). As a design experiment, this study reports on an experimental curriculum engineered to promote specific problem solving behaviors for the purpose of enriching and unifying PSMTs’ conceptual knowledge of mathematics and fostering mathematical practices that have been shown by research to be productive for mathematicians. Two case studies of PSMTs are presented, investigating their behaviors and subsequent growth as a result of their participation in the experiment. This investigation builds upon previous research into the problem solving behaviors of mathematicians, and extends this research into the arena of undergraduate teacher preparation. The Multidimensional Problem Solving Framework guided the design of tasks and activities in the instructional sequence and also provided the theoretical perspective for retrospective analyses of the data. The framework was used to identify evidence of the cycles of problem solving, use of resources, affect management, monitoring, and beliefs as PSMTs progressed through the course. Both of the students studied experienced improvement in their problem solving behavior and significant shifts in their views about mathematics. The quality of PSMTs’ conceptual understanding of the mathematics at hand was revealed to greatly impact the solvers’ success in constructing viable solutions. Well-connected conceptual understanding of mathematics appears to be a necessary but not sufficient condition for problem solving success.
DEDICATION

I dedicate this dissertation to my husband Jerry Bloom, who has always supported my academic journey. Thank you for making it possible.
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CHAPTER ONE
INTRODUCTION

This dissertation reports on one effort to improve the mathematical preparation of prospective secondary mathematics teachers (PSMTs). As a design experiment (Kelly & Lesh, 2002), this study reports on an experimental curriculum engineered to promote specific problem solving behaviors for the purpose of enriching and unifying PSMT’s conceptual knowledge of mathematics and fostering mathematical practices that have been shown to be productive for mathematicians (Carlson & Bloom, 2005). Two case studies of PSMT’s are presented, investigating their behaviors and subsequent growth as a result of their participation in the experiment.

This investigation builds upon my previous research collaboration with Dr. Marilyn Carlson into the problem solving behaviors of mathematicians, and extends this research into the arena of undergraduate teacher preparation.

Statement of the Problem

Experts in the field of mathematics education agree that prospective high school mathematics teachers need a substantial amount of mathematics beyond content at a secondary mathematics level (Conference Board of Mathematical Sciences, 2000; Ferrini-Mundy & Findell, 2001; H. Wu, 1999). The Teaching Principle in the National Council of Teachers of Mathematics’ Principals and Standards (2000) states that mathematics teachers need knowledge of “the whole mathematical domain; deep, flexible knowledge about curriculum goals and about important ideas that are central to their grade level” (p 17). The RAND Corporation asserts that the quality of classroom
instruction is dependent on the teacher and “what the teacher can do depends on their knowledge of the mathematics (RAND Mathematics Study Panel, 2003, p. xvi).

However, there is less agreement as to what kind of mathematical experiences prospective teachers need in order to be effective teachers. Traditionally, preservice high school teachers earn a degree in mathematics or a related topic. Yet those very students often do not see the relevance of requisite upper division mathematics courses, and do not recognize how the topics in courses such as Real Analysis and Abstract Algebra relate to school mathematics (Bloom, 2004; Conference Board of Mathematical Sciences, 2000).

The subject matter development of teachers is a fairly new strand of research inquiry (Ball & McDiarmid, 1990), and there exist many more studies about what mathematics teachers should know and what they do know than about how to bridge the gap between the two (e.g., Ball, 1990; Bryan, 1999; Conference Board of Mathematical Sciences, 2000; Koirala, 2002; McDiarmid, Ball, & Anderson, 1989). Undergraduate preparation generally goes well beyond the scope of what is taught at the K-12 level, yet rarely do students have the opportunity to revisit the topics that make up the secondary mathematics curriculum from a new perspective, or connect new mathematical information with school mathematics. Until recently, the idea that the concepts comprising high school mathematics may also comprise an appropriate course of study for prospective mathematics teachers was overlooked. Rather, there is an assumption that teachers will learn what they need to know about these topics in the process of teaching them; however, this assumption is not well supported by research.

There is agreement that mathematics teachers need a deep, well-connected knowledge base (Ball & Bass, 2000; Carpenter, Franke, & Levi, 2003; Fennema &
Franke, 1992; Ferrini-Mundy & Findell, 2001; Koency & Swanson, 2000; Ma, 1999).

The Conference Board of the Mathematical Sciences (CBMS) published a comprehensive report on the mathematical education of teachers (2000). This report proposed explicit curriculum guidelines for the development of mathematics teachers and made specific recommendations for both colleges of education and departments of mathematics. For example, Recommendation #1 states, “Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach;” Recommendation #3 states, “Courses on fundamental ideas of school mathematics should focus on a thorough development of basic mathematical ideas. All courses designed for prospective teachers should develop careful reasoning and mathematical ‘common sense’ in analyzing conceptual relationships and in solving problems.” Recommendation #4 says, “Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching” (Conference Board of Mathematical Sciences, 2000, p. 8). These recommendations from the CBMS indicate that prospective teachers do not necessarily need more mathematics; rather they need a better understanding of mathematics.

When discussing mathematics curriculum, the National Council of Teachers of Mathematics (NCTM) states:

Mathematics comprises different topical strands, such as algebra and geometry, but the strands are highly interconnected. The interconnections should be displayed prominently in the curriculum and in instructional materials and lessons. A coherent curriculum effectively
organizes and integrates important mathematical ideas so that students can see how the ideas build on, or connect with, other ideas, thus enabling them to develop new understandings and skills. (National Council of Teachers of Mathematics, 2000, p 14)

The document goes on to state that “To be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks” (p. 17). To establish the types of classrooms that are described in the *Principles and Standards*, teachers need to see the overall structure of school mathematics, see how the various content domains are related to one another, and be able to make connections to the “real world.” In order to do this, they need a deep understanding of the secondary curriculum – a curriculum in which most of them excelled, but few possessed a strong conceptual base of its topics.

Research into mathematical problem solving has suggested that powerful problem solvers tend to have a well-connected web of mathematical knowledge; they are flexible rather than rigid in their thinking they are reflective, and they monitor their work. They also demonstrate confidence, perseverance, and the ability to acknowledge and manage frustration, and they value sense-making in the pursuit of a solution (Arcavi, Kessel, Meira, & Smith, 1998; M. Carlson & Bloom, 2005; M. P. Carlson, 1999b; Santos-Trigo, 1998; A. Schoenfeld, 1992; A. H. Schoenfeld, 1985b). Thus, mathematics educators wish to cultivate these characteristics and ways of thinking in mathematics teachers.

*Motivation*

As a high school teacher, I came to understand that even though I was a high achiever as a mathematics major and an education minor, I still had serious gaps in my
mathematical knowledge. As an undergraduate, I had not thought carefully about the mathematics I was going to teach. When I began to teach, I had to work hard to explain simple concepts, and many times I struggled to solve problems in my students’ textbooks. Fortunately, I became involved with competition mathematics. To prepare for competitions, my students and I would practice solving problems in all domains of the curriculum. I came to see that this experience made it easier for me to explain concepts, point out connections across topics, and unpack mathematics from problems for all my students.

When I was hired as a lecturer by Arizona State University just after completing a master’s degree in mathematics, I was fortunate enough to start working under Marilyn Carlson on the task of improving mathematics teacher education in the mathematics department. I began to work with prospective elementary teachers, creating hands-on tasks to help them develop conceptual tools as well as curiosity about the mathematics they would be teaching. Not only was this extremely rewarding, but it was educational as well. I came to understand basic mathematics in new and interesting ways. For example, I began to consider the connections between algebra and arithmetic and use those connections in the college algebra classes I was also teaching to help my students see the abstraction of arithmetic in many of the algebraic concepts in the curriculum.

I also interviewed and observed prospective secondary mathematics teachers (PSMTs) who were either student-teaching or preparing to student-teach. The goal of these interviews was to assess and improve the preparation of high school teachers. I was truly surprised by what I found and heard. The preservice teachers I interviewed did not see a relationship between the upper division mathematics they took at the university and
the high school curriculum. They often remarked that they wished they could take a "refresher course" on high school topics, yet they also stated their confidence in knowing the mathematics well enough to teach it. When I observed their classrooms, the format was primarily lecture. The format was generally the same: a review of homework, short lecture, examples, seat work, and homework. Students were rarely engaged in trying to make sense of the mathematics.

Shortly after that experience, Dr. Carlson and I began to study the mathematical behaviors of mathematicians while solving non-routine problems. As I interviewed mathematicians and began to analyze the data, I was struck by several things. One was the breadth of mathematical knowledge many had at their fingertips and the variety of places that knowledge led. Although we used the same set of problems for each interview, the solution paths were distinct and individual. The mathematicians were articulate in their thinking, highly emotional, and very motivated. Not one asked if he arrived at the correct answer. They knew from reflecting on their own thinking and work if they were correct. When asked how they became good problem solvers, most credited solving many problems and having a good working knowledge of mathematics. The results of this research project have been published and will be discussed in more detail in Chapters Two and Three. In general, the experience impacted how I thought about my own teaching, problem solving, and mathematical understandings. It occurred to me that the problem solving behaviors and dispositions observed in these experienced mathematicians (curiosity, integrity, reflection, and monitoring, to name a few) would be useful ways of thinking and acting for mathematics teachers as well.
In another experience, I was asked to teach a course called "Content in Secondary Schools." This is a required mathematics education course for PSMTs. The course was not intended to be one of the methods of teaching courses, but rather a course that was designed to focus on mathematics topics in the high school curriculum. Although it was often referred to as "the problem solving course," it took many forms depending on who taught it. I saw this as an opportunity to design and teach a course that would require students to revisit secondary school topics, draw connections between topics, and deepen conceptual understanding of mathematics in general. I could purposefully cultivate the mathematical practices revealed in Carlson and Bloom’s (2005) problem solving research. This idea was the beginning of a five year design-based research project, the results of which are reported in this document.

The Research Questions

In response to national calls to improve the mathematical education of teachers, the following research questions guided my inquiry:

1. What changes were evident in the problem solving behaviors of prospective high school mathematics teachers as they progressed through an instructional sequence that engaged PSMTs in the practices of problem solving, mathematizing, generalizing, and extending?

As a design experiment, this study describes an instructional innovation employing thought revealing activities and artifacts in conjunction with inquiry into theoretical considerations (Kelly & Lesh, 2002; Lesh, 2002), in this case regarding the nature of the development of problem solving behaviors. In order to both describe and explain PSMT’s development, two case studies are provided: the Case of Amy is found
in Chapter Six, and the Case of Ben is found in Chapter Seven. The Multidimensional Problem-Solving (MPS) Framework is used to investigate and describe transformations in their problem solving behaviors and dispositions.

2. What changes in other personal knowledge factors – confidence, views about mathematics, self efficacy, persistence etc., — are evident as PSMTs progress through an instructional sequence that engaged them in the practices of problem solving, mathematizing, generalizing, and extending?

The MPS Framework is used to investigate changes in the personal knowledge factors described above.

3. In what ways did the instructional sequence influence changes in the problem solving behaviors and personal knowledge factors of PSMTs?

Chapter Five describes the instructional sequence that served as the curriculum for the course. Case studies provide analyses of the subjects’ interactions with the instructional sequence.

An Overview of the Study

A review of the literature that informed and guided the present study is found in Chapter Two. Chapter Three provides a discussion of the theoretical considerations and frameworks used to guide the development of the instructional sequence and the data analysis. A description of the methods used in this investigation is found in Chapter Four. Chapter Five provides a discussion of the instructional sequence, including results of prior iterations of the curriculum. Case studies of Amy and Ben are found in Chapters Six
and Seven respectively. A summary of the findings and discussions of the implications of this research are provided in Chapter Eight.
CHAPTER TWO
RELEVANT LITERATURE

This dissertation project draws from the rich and diverse body of work in the field of mathematics education. In particular, in this chapter I explore relevant literature in the areas of human cognition and what it means to know and understand mathematics, teacher knowledge and beliefs, problem solving, and problem solving instruction.

The first section in this chapter provides a discussion about literature in the area of cognitive science that informs the present study. It is important to note that the field of cognitive science offers useful insight into what it means to know and understand mathematics as it relates to problem solving. In this section I discuss the literature about knowing and understanding mathematics from a cognitive science perspective as well as the other theoretical stances that have emerged from this field. In the subsequent section, I review research on what teachers know and what they need to know in order to be effective in the classroom. Research in teacher knowledge — in particular, teacher knowledge of mathematics — highlights the distinction between simply knowing mathematics, and knowing mathematics for teaching. Following this discussion, I then examine various problem solving frameworks used to analyze the problem solving process. In general, problem solving research opens a window into how mathematical knowledge is (or is not) accessed and applied to resolve mathematical situations. In this section I discuss, in particular, the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005) that I used in the present study. Finally, this chapter concludes with a section on literature pertaining to problem solving instruction.
Cognition and Cognitive Frameworks

The study of cognition with respect to mathematics education has undergone an evolution as both fields have advanced in their thinking and research methodologies (Schoenfeld, 1987a). Today, most studies of cognition are grounded in one of two perspectives – the acquisition perspective and the participation perspective (Sfard, 1991). In the first perspective, knowledge is something that one acquires, and the focus is on internal mental mechanisms. In the second perspective, knowledge is “situated” in activity and the primary focus is on participation in social practices. There is a third perspective called the emergent perspective (Cobb, 2000) where both the acquisition (individual) perspective and the participation (social) perspective are seen as reflexively related, and, therefore, both ought to be investigated concomitantly.

To discuss cognition in terms of mathematics and mathematics education, it is important to be clear about what we mean by knowing and understanding mathematics. Hiebert and Carpenter (1992) define understanding:

…in terms of the way information is represented and structured. A mathematical idea, procedure, or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is a part of a network of representation (p. 67).

They describe two types of mathematical knowing; conceptual and procedural knowledge. Conceptual knowledge is defined as knowledge that is rich and connected (Hiebert & Lefevre, 1986). A richer network of connecting relationships signifies a richer quality of the conceptual knowledge. When previously unrelated networks become linked, cognitive reorganization has taken place. Procedural knowledge, on the other
hand, consists of two components: formal or symbolic representation systems and algorithms. Both forms of mathematical knowing contribute to the learning and practice of mathematics. Both help lessen the cognitive load, thus freeing resources to be used in sense making and problem solving. The connectedness of conceptual understanding offers broad access to mathematical concepts, while procedural knowledge allows for efficient processing.

Vinner (1997) suggests another kind of learning which is neither purely conceptual nor procedural. Pseudo-conceptual behavior may appear on the surface to be conceptual behavior, yet it lacks an associated connected network. “In mental processes that produce conceptual behaviors, words are associated with ideas, whereas in mental processes that produce pseudo-conceptual behaviors, words are associated with words; ideas are not involved” (p. 101). Likewise, while students are expected to be analytical, often students display behaviors that result in a correct answer but are pseudo-analytical in nature. Pseudo-analytical and pseudo-conceptual behaviors are characterized by poor self-monitoring and the absence of reflection on processes.

Mason and Spence (1999) suggest that factual and conceptual knowledge is not sufficient for success in mathematics and mathematical problem solving. One can have the requisite knowledge base to solve a problem yet not “know to-act-in-the-moment.” “The purpose of distinguishing knowing-to from other kinds of knowing is that it is precisely the absence of knowing-to which blocks students and teachers from responding creatively in the moment” (p. 143).

According to Mason and Spence (1999), one must first have a sense of not knowing-to. Knowing-to can be characterized as a sudden shift of attention or focus.
“Once the moment of knowing-to takes place, knowing-how takes over to exploit the fresh idea” (p. 146). However, repeated practice is an inadequate foundation for knowing-to, but intentional reflection on past actions can prepare students to know-to-act. Reflective practices provide the means to collect an array of possible actions and develop a kind of awareness or noticing that is necessary to recognize critical moments.

The previously discussed theories of mathematical knowing are focused on individual mental activities. Taking a situated perspective, Lave and Wenger (1991) claim that learning takes place in the context of activity and community. “Learning viewed as situated activity has as its central defining characteristic a process that we call legitimate peripheral participation” (p. 29) in a community of practice. A community of practice is loosely defined as a community including persons involved at various levels of participation, activities that are carried out by the community, associations within the community members and the world, and a system by which the community reproduces itself (Lave & Wenger, 1991). Participating in the community of practice and moving toward full participation (i.e., learning) also entails coming to identify oneself as a member of the community. From this perspective, the primary motivation for learning is fuller participation in the community and the construction of identity as a member of the community. Legitimate peripheral participation is a way for newcomers to a community of practice to become full participants in that community. Lave and Wenger suggest that lack of student motivation is a failure of traditional didactic schooling. The learner becomes something to be acted upon rather than a co-participant, and so the learner experiences conflict between learning to know and learning to demonstrate knowing for testing.
This theoretical perspective has important implications for the study of learners’ conceptual shifts and conceptual growth in mathematics. As the theoretical grounding of cognition moves from acquisition to participation, the focus of inquiry moves from “inside the head” to activity within the community. In the later, evidence of learning would be marked by changes in the quality of the interaction within the community in question. This shift in theoretical perspective also necessitates changing research contexts from one-on-one interviews to classroom or other group observations.

It is from this situated perspective that Gravemeijer et al. (2000) and others involved in Realistic Mathematics Education (RME) invoke the concept of “guided re-invention” as a way of abstracting mathematical concepts. RME is a theoretical approach to teaching and learning mathematics that intertwines theory about how children learn mathematics with theory about how mathematics needs to be taught. In the tradition of RME, students should be offered opportunities to invent or create some of the fundamental concepts in mathematics. Emphasis in the classroom should therefore be placed on mathematizing activities rather than memorization of algorithms and procedures. From this theoretical perspective, effective activities start with a problem situation that is “real” for the students and allows students the opportunity to progressively mathematize the situation in increasingly abstract ways, successfully moving from a concrete representation to formal reasoning (Gravemeijer, 1994; Gravemeijer et al., 2000).

The theories discussed in this section provide insights into learning and understanding mathematics. Since the participants in this study were prospective high
school mathematics teachers, the next section moves from a discussion of cognition and
cognitive frameworks to that of teacher knowledge.

Teacher Knowledge

Research in teacher knowledge, teacher knowledge of mathematics in particular,
highlights the distinction between simply knowing mathematics and knowing
mathematics for teaching.

Shulman (1986) defines three specific kinds of knowledge that teachers need in
order to be effective: content knowledge, pedagogical content knowledge, and curricular
knowledge. Content knowledge encompasses the structure and organization of
knowledge in the discipline.

Teachers must not only be capable of defining for students the accepted
truths in a domain. They must also be able to explain why a particular
proposition is deemed warranted, why it is worth knowing, and how it
relates to other propositions, both within the discipline and without, both
in theory and in practice. (Schulman, 1986, p. 9)

Pedagogical content knowledge, on the other hand, refers to the intersection of domains
of content knowledge and teaching knowledge. It includes an understanding of what
makes a topic difficult to understand, what kinds of misconceptions students might
develop, and what sorts of strategies are most appropriate. Curricular knowledge is yet a
third kind of teacher knowledge: knowledge of the full range of curricula and materials
available for that subject or topic. Teachers need to know what the students were
expected to have learned before they arrived, and what they will need to learn in order to
be successful at the next level. Teachers also need to know how to appropriately use the
tools of the trade. In the case of secondary mathematics, that would include knowing how to use graphing calculators and dynamic geometry software.

Research into the affective domain of teacher knowledge is also informative. Teachers model attitudes as well as conceptions of mathematics (Post, Harel, Behr, & Lesh, 1991). Beliefs and other affective components are known to have a direct influence on classroom decision-making, and the capacity for change and reform in individual practice (cf Berdot, Blanchard-Laville, & Bronner, 2001; Cooney, Shealy, & Arvold, 1998; A. Thompson, 1992; Vacc & Bright, 1999). In particular, as modelers of problem solving behaviors, practitioners need to be reflective about their own mathematical behavior and in turn teach their students metacognitive skills in addition to technical proficiency (A. Thompson, 1992).

In light of recent reports about performance in mathematics as measured by national and international tests, there have been several concerted efforts to delineate what teachers need to know in order to teach mathematics effectively (Conference Board of Mathematical Sciences, 2000; Mathematical Sciences Education Board, 2001; National Science Foundation, 1998; RAND Mathematics Study Panel, 2003; The Content Panel for Mathematics, 2002). In *The Mathematical Education of Teachers*, the Conference Board of the Mathematical Sciences (2000) strongly advocates that preservice high school math teachers take a course devoted to the high school curriculum. This course should offer prospective teachers an opportunity to revisit high school algebra and geometry after taking upper division theoretical coursework. One answer to this call is the text *Mathematics for High School Teachers: An Advanced Perspective* (Usiskin, Peressini, Marchisotto, & Stanley, 2003). In this curriculum, the topics found in
high school are extended into mathematics of a more theoretical nature. Zazkis (1999) recommends that preservice teachers engage in experiences that challenge their basic assumptions about mathematics in order to construct richer conceptualizations of mathematics. These are just two of many efforts to find ways to enhance teachers’ conceptual understandings of high school topics. The present study will describe another such effort.

Various research studies show that both in-service and preservice teachers have been inadequately prepared in the past (Arizona Board of Regents, 1995, 1997; Ball, 1990; Bloom, 2001; Bryan, 1999; Luft, Buss, Ebert-May, & Eslamieh, 1997; Post et al., 1991; Tirosh & Graeber, 1990; U. S. Department of Education, 2002). Collectively they demonstrate that both elementary and high school in-service and preservice teachers have a shallow and fragmented knowledge base. If the teachers evidence procedural proficiency, they often fail to demonstrate conceptual proficiency. And it has been reported that poor conceptual understanding of mathematics can derail efforts to reform mathematics teaching and curriculum (Koency & Swanson, 2000).

In an exhaustive literature review, Thompson (1992) noted that it is nearly impossible to study teacher knowledge without accounting for a teacher’s beliefs. One reason is that a teacher’s beliefs greatly influence how the teacher’s knowledge is held and valued. Another is that teachers treat their beliefs as knowledge. Finally, the literature review found that beliefs about mathematics and mathematics teaching play a critical role in how a teacher shapes instructional activities and makes instructional decisions. Therefore, a teacher who perceives mathematics as fluid, dynamic, and problem driven
will devise different classroom interactions than a teacher who sees math as a static and complete body of knowledge.

The studies examined in Thompson’s (1992) review show that teachers’ belief structures are surprisingly robust and that they can hold contradictory beliefs about mathematics teaching and learning without a lot of discomfort. The results of efforts to change teachers’ beliefs were mixed.

The inconsistencies reported in these studies indicate that teachers’ conceptions of teaching and learning mathematics are not related in a simple cause and effect way to their instructional practices. Instead they suggest a complex relationship with many sources at work; one such source is the social context in which mathematics teaching takes place with all the constraints it imposes and all the opportunities it offers (p. 138).

One common thread in studies investigating teachers’ beliefs is that teachers are most willing and able to assimilate new points of view when they are intrinsically dissatisfied with their teaching practices. Cooney, Shealy and Arvold (1998) tracked the beliefs of four preservice secondary teachers as they moved through their coursework and student teaching. Their analysis identified three categories: initial understandings about mathematics and teaching, the search for affirmation of initial and developing beliefs, and beliefs at the time when they looked forward to the student teaching experience. While all four participants experienced conflict regarding their beliefs, some were able to resolve the conflicts by adjusting their points of view, but others were unwilling to modify their belief structures. The authors advocate that teacher preparation programs take into
account the belief structures of their students because “…accounting for the structure of beliefs enables us to create activities that encourage teachers to wonder, to doubt, to consider what might be, to reflect, and most important, to be adaptive” (p. 332).

Other studies have shown how robust a belief system can be. In a case study of a prospective mathematics teacher as she progressed through her undergraduate program in education and out to the classroom, Benken and Wilson (1996) showed that the individual’s core beliefs changed very little during that time. Although she was immersed in student-centered teaching in her methods classes and her internship and demonstrated that she could teach in innovative ways, she continued to hold traditional views about teaching and learning mathematics.

Millsaps (2000) used autobiographies as a tool for understanding the teaching and learning beliefs of mathematics teachers. She found that personal experiences, both positive and negative, played important roles in defining mathematics and teaching for the teachers she studied. The author suggests that the kind of experiences needed to change teachers’ beliefs need to be just as significant and just as personal and must confront their existing beliefs. Berdot (2001) reported similar findings. He suggests that the curriculum changes implemented in light of reform efforts can be highly threatening to teachers’ self-images as educators and mathematics experts, and he cautions that those implementing reforms ought not to underestimate the level of conflict these changes can cause the instructors who are expected to adjust. It is possible, nonetheless, to successfully facilitate teacher change (Ball, 1988; Benken & Wilson, 1996; Fryholm, 2000; Kaput & Blanton, 1999; Kinach, 2002; Lachance & Comfrey, 2003; Zazkis, 1999). One example of teacher change is the Cognitively Guided Instruction project (Carpenter,
Fenema, Peterson, Chaing, & Loef, 1989). In the Cognitively Guided Instruction project, rather than learning specific teaching strategies the teachers were immersed in trying to understand how children made sense of certain arithmetic tasks. As a result, a majority of these teachers substantially changed their teaching of addition and subtraction. Their students engaged in problem solving significantly more than the control groups, and these teachers demonstrated that they knew more about the thinking processes of their students than the other group. Their students scored higher on measures of proficiency, and there was a measurable shift in the teachers’ beliefs about mathematics teaching and learning.

In the next section, I show that like mathematics teachers, problem solvers, require a deep, well-connected, flexible, and fluid knowledge base in order to be successful.

*Problem Solving Frameworks*

Much of the research on problem solving and mathematical thinking has been directly influenced by mathematician George Polya and his book, *How to Solve it* (Polya, 1957). In the introduction to the book, Polya stresses the importance of allowing students the experience of solving mathematical problems on their own so that they may “experience the tension and enjoy the triumph of discovery” (p. v). He says that in solving mathematical problems, we are introduced to the “other face of mathematics … [the part that is] in the process of being invented” (p. vii).

This work was intended as a “how to” text for teachers of mathematics as well as people who enjoyed recreational mathematics. He was interested in sharing his wisdom and experience as a mathematician and teacher with a wider audience. It is in this context that he first introduced his well known framework: Polya’s 4-steps of problem solving.
These steps are: *understanding the problem, devising a plan, carrying out the plan, and looking back*. In this last step, Polya not only advises that one check the validity of the solution, but suggests that one see if the problem can be solved by a different method, or if a new problem can be created from the original one. He points out that many students miss a valuable learning opportunity by not gleaning as much from the problem as possible. He also points out that teachers miss the opportunity to have students investigate the connections that that problem has with other mathematics. Polya (1957) introduces the term *modern heuristics* to describe the strategies or the “mental operations typically useful for the solution of problems” (p. 2).

While this framework was based on Polya’s personal experiences as a mathematician and teacher rather than as a formal educational researcher, it still serves as a useful foundation for thinking about problem solving today. Polya’s 4 steps for problem solving, in one version or another, appear in many secondary mathematics texts as a guide to problem solving. Additionally, most all research reports on aspects of problem solving include a discussion of Polya’s work.


Schoenfeld’s (1985a) experience with teaching problem solving led him to formally research what was involved in the process. His search resulted in a framework
for analyzing the knowledge and behaviors required for successful problem solving. He asserts that specialized knowledge and behaviors are required for competent problem solving performance. He has named these: resources, heuristics, control, and belief systems. According to Schoenfeld, resources encompass all the mathematical knowledge the solver brings to the table. This includes (but is not limited to) intuitions, definitions, algorithms, conceptions, and misconceptions. Heuristics are defined much as Polya defined them – techniques employed to make progress on non-routine problems. Control refers to the global decisions the solver makes along the way. These decisions can include: which resources to access, which strategies to employ, on-line monitoring, and metacognitive decisions. Belief systems represent one’s “mathematical world view” (p. 15). These are the individual’s beliefs (both conscious and unconscious) about one’s own abilities, the problem situation, or mathematics in general.

When discussing resources, Schoenfeld (1985a) notes that expert problem solvers possess a rich conceptual schema for mathematical problem solving as well as considerable experience so that certain problem situations will evoke certain productive responses. In contrast, novices not only lack such a schema, but their schema often contains misconceptions and misinformation. Moreover, many students exhibit consistent error patterns that stem from making inaccurate connections.

Drawing from Polya, Schoenfeld sees heuristics as a key aspect of mathematical thinking. In examining how experts approach problem situations, he noticed a great deal of commonality in the methods they employed. Yet he found that simply teaching heuristic strategies is insufficient (Schoenfeld, 1985a). Having appropriate resources and implementing problem solving strategies are only part of the picture. Hundreds of
decisions are made in the course of solving a problem. A poorly managed solution path will generally end up in failure. Therefore, metacognition is required. Schoenfeld (1992) defines metacognition as “(1) individuals’ declarative knowledge about their cognitive processes, (2) self-regulatory procedures including monitoring and “on-line” decision-making, and (3) beliefs and affects and their effects on performance” (p. 3). The category control encompasses the first two aspects of metacognition along with the cognitive acts of knowing and reasoning.

Schoenfeld (1985a) found that belief systems have tremendous influence over the problem solving process. For example, if a student believes that she is not a good problem solver, this belief influences her work, as does the belief that the solution does not need to make sense, or that the problem is difficult or easy. “Beliefs establish the context in which resources, heuristics and control operate” (p. 45). Hence, all of these aspects, as well as the solver’s mathematical practices, play a part in the problem solving process.

Building on the work of Schoenfeld and Polya, a number of mathematics education researchers have also created frameworks for examining mathematical problem solving. For example, Lester, Garafolo, and Kroll (1989b) defined four cognitive and non-cognitive factors that influence problem solving (affects and attitudes, beliefs, control, and contextual factors), as well as four categories of activities that are involved in attempting a mathematical task (orientation, organization, execution, and verification). Unlike the Schoenfeld framework, this framework assumes that the subjects have the appropriate knowledge needed to solve the problem. They define control as knowledge
about cognition and regulation of cognition, and they separate attitudes and beliefs into two separate categories.

Geiger and Galbraith (1998) were primarily interested in investigating the behavior of secondary mathematics students when solving application-type exercises in ordinary classroom situations (as opposed to a laboratory-type setting). Their framework was developed to help them analyze students' written solutions. The researchers found that high quality responses in the planning and monitoring area almost always followed equally high quality representations in the initial engagement, suggesting the importance of initial engagement with the problem. They also found that heuristics were not really used, with the exception of the use of diagrams, even though these students had been instructed in the use of heuristics. They also noticed a general lack of evidence of verification across the board. The authors hypothesized that a student’s belief that the right answer is more important than the process may have contributed to the lack of sense making and verification skills, as well as to the practice of either altering or ignoring important aspects of the original problem statement.

The Multidimensional Problem-Solving Framework developed by Carlson and Bloom (2005) emerged from the desire to capture the complex interaction of cognitive, metacognitive, and non-cognitive behaviors observed as experienced mathematicians solved mathematics problems. The framework is sufficiently detailed to allow researchers to tease out specific aspects of mathematical behavior. This framework is the lens through which I studied the mathematical practices, beliefs, and attitudes of the students in the present study. I provide a more detailed description of this research and its implications for this project in Chapter Three.
Goos, Galbraith, and Renshaw (2000) were particularly interested in the self-monitoring and subsequent regulation of high school students as they attempted to solve mathematics problems. They distinguish *routine monitoring* from more controlled actions that a student takes when he realizes that he is experiencing difficulty in solving problems. The authors referred to the latter as these *metacognitive red flags*. Three types of red flags were identified: lack of progress, error detection, and anomalous result. They found that when students were unsuccessful, one of the following events occurred: *anomalous results* were verified and accepted; *lack of progress* did not lead to a change in strategy, or *errors* went undetected. The researchers concluded that if teachers want students to monitor their progress in mathematical situations, students should receive instruction in both the recognition of red flags and the actions that they should signal.

There has also been a research effort to understand how affective factors influence problem solving behavior. DeBellis (1998) and Goldin (DeBellis & Goldin, 1991, 1997, 1999) investigated the interaction between affect and cognition. Whereas researchers such as Schoenfeld focused on *global affect* (such as attitudes and beliefs), DeBellis and Goldin looked instead at the influence of *local affect* on problem solving. Local affect is defined to be “the changing states of feeling during problem solving” (DeBellis & Goldin, 1991, p. 29) including puzzlement, curiosity, frustration, and satisfaction. In particular, *mathematical intimacy* and *mathematical integrity* were found to be traits of powerful problem solvers (DeBellis, 1998; DeBellis & Goldin, 1999). According to DeBellis (1998), mathematical intimacy is characterized by an emotional bond between the solver and the mathematics content, and exhibited behaviors may be expressions of joy or anger, cradling the work, and genuine excitement. Mathematical integrity refers to
the solver’s sense of whether the solution is correct or justified and a willingness to admit mathematical shortcomings.

Local affect appears to influence problem solving in important ways by encouraging engagement (or disengagement) with the problem, shaping the selection of strategies and solution paths, and promoting verification. However, affect can be a negative force as well, derailing the less competent or confident students. “We have seen examples in which affect appears to guide problem-solving choices and where powerful problem solvers use it effectively. However, affect can also have negative consequences, even in strong students” (DeBellis & Goldin, 1991, p. 35).

The next section provides a discussion of problem solving instruction.

*Problem Solving Instruction*

Problem solving instruction is another area of interest. The National Council of Teachers of Mathematics (NCTM) (2000) has identified mathematical problem solving as being central to the study of mathematics.

Problem solving is an integral part of all mathematics learning. In everyday life and in the workplace, being able to solve problems can lead to great advantages. However, solving problems is not only a goal of learning mathematics but also a major means of doing so. Problem solving should not be an isolated part of the curriculum but should involve all Content Standards (p. 52).

The NCTM’s (1989, 2000) call for increased focus on mathematical problem solving led to increased interest in the instructional practices that promote mathematical problem solving (Charles, 1985).
However, in many textbooks, problem solving instruction is limited to Polya’s 4 steps, followed by a list of useful problem solving strategies such as “draw a diagram” or “work backwards” (for example, see Billstein, Libeskind, & Lott, 2004) and a few story problems at the end of the chapter. By contrast, other texts are devoted entirely to problem solving. For instance, *Crossing Rivers with Dogs* (Johnson & Herr, 1994) (geared toward middle and high school) is comprised of non-routine problems and puzzles divided into sections that practice specific heuristics. Some mathematicians and mathematics educators (myself included) would take exception to this segregation of problem solving in the school curriculum. Wu (1997) points out that

Among educators, there is a tendency to isolate problem solving as a separate component of mathematics education and cultivate it per se. However, mathematicians take a different stand on the issue: they place problem solving right in the context of content knowledge (p. 15).

According to Skemp (1993), while routine methods may be useful for solving routine problems, they lack the flexibility and adaptability needed for true problem solving. The best way to help students become good at problem solving involves taking them via a detour, that of building up necessary knowledge structures. Once they have these, they will be equipped to solve a number and variety of problems far greater than could be possible from any direct approach. (p. 3).

Santos-Trigo (1998) identifies three approaches to problem solving instruction: *Problem Situations*, the *Moore Method*, and *Schoenfeld’s Approach*. Instruction that follows the Problem Situation approach makes use of well selected problems as the
motivation and the means for learning mathematical concepts. The Moore Method refers to the teaching style of R. L. Moore, the noted topologist from the University of Texas. Moore began each semester by providing students with the basic definitions and axioms of the topic. He then asked students to prove all the theorems by themselves and present their proofs in class. During their presentations he queried them to justify their solution approaches. The Schoenfeld Approach is modeled upon Schoenfeld’s Problem Solving Course, where problem solving is the course content, and deeper understanding of mathematical content is a happy by-product.

Analysis of problem solving instruction has been conducted on Schoenfeld’s problem solving course at UC Berkeley. In his extensive writing about the course, Schoenfeld (1982, 1983, 1985a, 1985b, 1987a 1992, 1995, 1998), used the course setting as a laboratory for two purposes: to study problem solving and to develop theory about the nature of mathematical problem solving and problem solving instruction. Other researchers (Arcavi et al., 1998; Santos-Trigo, 1998) have also observed his classes and reviewed his data for their own work. Santos-Trigo (1998) investigated how Schoenfeld engaged his students in problem solving as well as how he motivated his students to explore mathematics beyond the original problem statement. He concluded that Schoenfeld created a classroom environment where students were expected to work on a variety of problems, discuss their solutions and strategies, reflect on their work and communicate it to others (both in written and oral form), participate in mathematical discourse, and search for connections and extensions to problems they had solved. Arcavi et al. (1998) add that Schoenfeld modeled the behaviors he expected of his students, and he was able to scaffold the use of heuristics, metacognitive strategies and problem
extensions in a way that allowed his students to have authentic problem solving experiences and learn these new skills.

Summary

Collectively, the research literature on topics relating to the field of mathematical problem solving suggests (a) myriad and intricate ways in which various cognitive and affective aspects influence problem solving and (b) identifiable components constituting the problem solving process. In this chapter, I have presented those studies that have most directly informed my dissertation project. The next chapter explicates the philosophical underpinnings that informed my study.
CHAPTER THREE
THEORETICAL FRAME

This chapter addresses the theoretical considerations of this research project. In the first section, I provide a description of the results of the problem solving study (Carlson & Bloom, 2005) that informed this research. This is followed by a discussion of the Multidimensional Problem-Solving (MPS) Framework and its use in this study both as a tool for designing mathematical tasks and a lens for analyzing my data. The second section provides a description of design-based research. This is followed by a discussion of my motivation for selecting this research paradigm for this study.

The Problem-Solving Framework

The MPS Framework, the primary investigative tool used in the present study, emerged from a study of mathematicians’ problem solving behaviors as they attempted to solve non-routine problems (Carlson and Bloom, 2005). The Multidimensional Problem-Solving Framework (Table 1) characterizes the complex interactions between the products and processes of mathematical problem solving. It describes problem solving as a cyclic process, with the problem solver’s (a) mathematical knowledge, (b) heuristics, (c) beliefs and emotions, and (d) metacognitive behaviors affecting the effectiveness of their solution approaches.

A Description of the MPS Framework

While analyzing the transcripts of the problem solving sessions, Carlson and Bloom noticed a regular pattern of cognitive and metacognitive actions. These phases were cyclic in nature, as the solver would run through the cycles until the problem was solved or abandoned. These phases are orienting, planning, executing, and checking...
According to Carlson and Bloom (2005), orienting includes the mental activities involved during the initial cognitive engagement stage of attempting to understand the problem. The exhibited behaviors include reading and rereading the problem, organizing the information, defining unknowns, sketching a graph or table, and any other behavior indicating that the subject is attempting to make sense of the problem statement.

Planning includes the mental activities involved in determining a solution approach. In this phase, a solution approach is considered, and the individual imagines how it will play out, while considering and then selecting or eliminating various strategies and tools. The exhibited behaviors include the statement of a conjecture and verbalization of a general solution approach. During this phase, individuals have been observed engaging in what Simon (1998) has called transformational reasoning. That is, the individual creates a system and informally tests it to get a feel for the correctness of her/his conjecture before proceeding.
<table>
<thead>
<tr>
<th>Phase</th>
<th>Behavior</th>
<th>Resources</th>
<th>Heuristics</th>
<th>Affect</th>
<th>Monitoring</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Orienting</td>
<td>• Sense making</td>
<td>Mathematical concepts, facts and algorithms are accessed when attempting to make sense of the problem. The solver also scans her/his knowledge base to categorize the problem.</td>
<td>The solver often draws pictures, labels unknowns and classifies the problem. (Solvers were sometimes observed saying, “this is an X kind of problem.”)</td>
<td>The curiosity and interest level of the solver affects the solver's motivation to make sense of the problem. If the solver is not interested, he/she often lacks motivation and stalls before starting.</td>
<td>Self-talk and reflective behaviors serve to keep the mind engaged. The solvers were observed asking “What does this mean?”; “How should I represent this?”; “What does that look like?”</td>
</tr>
<tr>
<td>• Planning</td>
<td>• Conjecturing</td>
<td>Conceptual knowledge and facts are accessed to construct conjectures and make informed decisions about strategies and approaches.</td>
<td>Specific computational heuristics were accessed and considered while considering and choosing a solution approach.</td>
<td>Beliefs about the methods of mathematics and one’s abilities influence conjectures and decisions. Signs of intimacy, anxiety, and frustration are also displayed.</td>
<td>Solvers monitor their strategies and plans. They ask themselves “Will this take me where I want to go?”; “How efficient will approach x be?”</td>
</tr>
<tr>
<td>• Executing</td>
<td>• Computing</td>
<td>Conceptual knowledge, facts and algorithms are accessed when executing, computing and constructing. Without conceptual knowledge, monitoring of constructions is misguided.</td>
<td>Fluency with a wide repertoire of heuristics, algorithms and computational approaches are needed for the efficient execution of a solution.</td>
<td>Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerge in the context of constructing and computing.</td>
<td>Conceptual understandings and numerical intuitions are employed to monitor both the solution progress and products while constructing statements.</td>
</tr>
<tr>
<td>• Checking</td>
<td>• Verifying</td>
<td>Resources, including well-connected conceptual knowledge informed the solver as to the reasonableness or correctness of the solution attained.</td>
<td>Computational and algorithmic shortcuts are used to verify the correctness of the answers and to ascertain the reasonableness of the computations.</td>
<td>As with the other phases, there were a number of affective behaviors displayed. It is often at this phase that frustration overwhelmed the solver, causing the solver to abandon the task.</td>
<td>Reflections on the efficiency, correctness and aesthetic quality of the solution provided useful feedback to the solver.</td>
</tr>
</tbody>
</table>

*Figure 1: The Multidimensional Problem-Solving Framework*
Executing includes the activities involved in solving the problem. The exhibited behaviors include writing logically connected mathematical statements, determining unknowns, execution of specific strategies, and carrying out computations.

Checking includes the mental activities involved in determining the correctness of the solution approach. The exhibited behaviors include spoken reflections on the product or process of the work, written computations to verify the correctness of the answer, and general remarks to verify that the answer is reasonable and fits with known information.

Monitoring refers to the mental actions involved in regulating the problem solving process. The exhibited behaviors include pauses in the execution phase to determine if the product makes sense or reflections regarding the productiveness of a solution approach (e.g., Is this approach getting me anywhere? What does this tell me?). It is important to note that checking and monitoring were seen to be two different processes. The checking phase generally followed the executing phase as the mathematicians verified the validity their work. Monitoring, on the other hand, occurred throughout the problem solving process as the mathematicians exercised on-line management and sense-making skills as they worked through the problem.

Rather than characterizing the entire solution path, these cycles (Figure 1) were often very tight as the subjects worked through the problem. That is, they would consider one step in the process of solving the problem and run through the cycle, and if the checking phase was satisfactory, they moved on to the next logical sub-goal. If not, they started over. The researchers also noticed that during the planning phase, it appeared that the subjects were actively sorting through various possibilities, playing them out in their
imagination, before selecting a plan. Often, the mathematicians were observed staring into space at this point in the solving process. When the subjects were asked to verbalize their thoughts during this process, they revealed another mode of cyclic thinking. They would make a conjecture, imagine the conjecture being played out in some way, and then decide if the idea was worth pursuing. The authors labeled this sub-cycle the conjecture-imagine-evaluate cycle (see Figure 2). Influencing all these phases were resources, heuristics, affect, and monitoring. The resulting matrix (Figure 1) delineates how each of these aspects is manifested in each phase of the problem solving process. This work validates and extends Schoenfeld’s work discussed in Chapter Two.

Using the MPS Framework to analyze the problem solving sessions, Carlson and Bloom (2005) observed that these experienced problem solvers drew from a rich network of mathematical knowledge. In addition, they engaged in continuous self-talk and monitoring that prevented them from pursuing non-productive problem solving behaviors. While the mathematicians had tremendous self-confidence, they displayed a wide variety of emotional responses such as frustration, joy, dread, and excitement. However, it appeared that they were adept at managing the emotional component of the work so that emotions enhanced rather than detracted from their engagement with the problem to be solved.
Figure 2: A flow chart representing the problem solving cycles and sub-cycles.
The MPS Framework in the Present Study

In the present investigation, I chose to use the Multidimensional Problem-Solving Framework for several purposes. The MPS framework was a very useful tool for designing tasks and activities in the instructional sequence. The framework provided benchmarks for desired mathematical behaviors and practices, as well as habits that experienced problem solvers routinely employed. In this intervention, not only was I interested in having students orient themselves to the problem, conjecture about solution paths, plan, execute, and check as they solved problems, but I also intended to foster the other aspects of problem solving detailed in the MPS Framework. Desired learning outcomes of instruction included evaluating mathematical resources, managing frustration, and the justification and verification of claims and conjectures. In this way, the framework guided the task creation and implementation.

The MPS Framework also provided the lens through which I performed a retrospective analysis of the data. For example, I compared students’ behaviors to the desired behaviors enumerated in the framework. This allowed for the documentation of the evolution of such behaviors over the course of the instructional sequence. Additionally, I was able to use the framework to look for evidence of the cycles of problem solving as well as use of resources, affect management, monitoring, and beliefs as students progressed through the course.

Design-Based Research Theory

Design-based research typically refers to iterative refinement of some instructional innovation employing thought revealing activities and artifacts in
conjunction with inquiry into theoretical considerations, in this case regarding the nature of the development of mathematical knowledge (Kelly & Lesh, 2002; Lesh, 2002). For this research project, I required a research design that would support my efforts to “attempt to engineer innovative educational environments and simultaneously conduct experimental studies of those innovations” (Brown, 1992, p. 141).

The term design is borrowed from engineering and other applied sciences because product design necessarily focuses on the development of an outcome while at the same time attending to theory building (The Design-Based Research Collective, 2003). Those who engage in this type of research assert that this approach bridges the gap between research and practice, blurs the distinction between researcher and practitioner, and honors the complexities of the questions to be studied in educational research today. Additionally, technological innovations in data collection and dissemination such as digital video, interactive software, and hypertext demand similar advances in research methodologies (Lesh, 2002).

The present study is a design experiment. Although this dissertation does not describe a classroom teaching experiment, I made use of many of the characteristics of classroom teaching experiments.

A primary purpose for using [classroom] teaching experiment methodology is for researchers to experience, first hand, students’ mathematical learning and reasoning. Without the experiences afforded by teaching, there would be no basis for coming to understand the powerful mathematical concepts and operations students construct or even for
suspecting that these concepts and operations may be distinctly different from those of researchers. (Steffe & P. Thompson, 2000, pp. 267-8)

The present design experiment is comprised of a sequence of teaching episodes involving a teacher, a group of students, and a means of recording each episode. I was the instructor for all 22 episodes in the instructional sequence, which was digitally videotaped by an independent videographer. These recordings are used both to inform the following episode(s) and to provide evidence for subsequent retrospective analyses and theory building (Steffe & P. Thompson, 2000).

One important component of the present experiment is the hypothetical learning trajectory. The hypothetical learning trajectory describes a possible path that students might follow in their learning and development of specific mathematical ideas, incorporating means of support and organization of this learning. According to Simon (1995):

The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students’ thinking and understanding will evolve in the context of the learning activities.

(p. 136)

Another critical component of this study is the retrospective analysis because a primary purpose of design experiment is to “place classroom events in a broader theoretical context, thereby framing them as paradigmatic cases of more encompassing phenomena” (Cobb, 2000, p. 326). By carefully analyzing the data corpus, the researcher
can begin to develop and test models of mathematical thinking and learning. Finally, design experiments are iterative in nature, so the various aspects of one cycle inform the refinement of the next cycle.

Preliminary Studies

Design-based research theory has informed and guided both the research design of this project and the classroom innovations that were implemented and refined. The present study describes the fifth iteration of this design experiment. Each iteration of the research cycle focused on different aspects of the prospective secondary mathematics teachers’ (PSMTs’) mathematical behaviors (such as problem solving behaviors and beliefs), making each iteration a pilot study. Thought-revealing activities were created, implemented, and honed during the process; the activities were carefully sequenced to help students cultivate conceptual understanding of high school mathematics.

The instructional sequence was increasingly refined to the point that by the end of the fourth iteration, selection and sequence of tasks was fairly stable. The prior iterations have informed the present study in the following ways: the order and composition of the problem sets, the sequencing and scaffolding of the mathematical tasks used in class, and the refinement of questions and mathematical problems used in the pre- and post-instruction interviews. In particular, early iterations alerted me to undergraduates’ deficiencies with respect to their problem solving abilities (Bloom, 2002, 2004a, 2004b). Chapter Five offers a detailed description of the instructional sequence.

My own participation in this ongoing design-based research project has been extensive and varied. I created all the problem sets as well as the classroom activities; I
taught all five iterations of the experiment and conducted the retrospective analysis described in this dissertation.

**Summary**

In this chapter, I have laid out the principal theoretical underpinnings for this dissertation study: the Multidimensional Problem-Solving Framework and the design-based research approach. The MPS Framework, in addition to the design experiment aspects of hypothetical learning trajectory and retrospective analysis, guided me in each research decision for the study. The methods of my study are explained in the next chapter.
CHAPTER FOUR

METHODS

This chapter provides a description of the methods for the study. The Multidimensional Problem-Solving (MPS) Framework informed both the research design and the data analysis. The first section provides an overview of the research design. This is followed by discussions of the subjects and settings and sources of data. The second section discusses the methods used for analyzing the data corpus.

Research Design

The research design for the present study was informed by the problem solving research previously conducted by Carlson and Bloom (2005). The intended goal of the present design-based research project was to develop and study curricula to foster the development of mathematical thinking and problem solving abilities in prospective secondary mathematics teachers (PSMTs).

I began the project by developing an instructional sequence I hypothesized would support PSMTs in developing coherent conceptual and procedural knowledge of high school mathematics, flexible thinking about mathematics and mathematical problem solving, and the habit of reflecting on and monitoring one's own thought processes. Another goal of the instructional sequence was to foster the development of mathematical dispositions such as the following: (a) confidence in one’s mathematical ability, (b) perseverance in constructing a solution to complex problems, (c) ability to acknowledge and manage frustration, and (d) use of sense-making in the pursuit of a solution. These ways of thinking and mathematical habits of mind were targeted because they were
determined to be so productive for the mathematicians in the Carlson and Bloom, (2005) study. This instructional sequence, which included problem sets, classroom activities, and Extended Analysis Tasks implemented over 22 classroom sessions that were designed and arranged specifically to foster these behaviors, is detailed in Chapter Six.

Setting

Next, I describe the course setting in which the instructional innovation was implemented. The course, Mathematics in the Secondary School, was required for all students seeking state certification to teach high school mathematics, and was considered a mathematics course as opposed to a “methods for teaching” course. The university course catalog description is as follows:

| MTE 483 Mathematics in the Secondary School. (3) | 
| Topics in geometry, number theory, algebra, and analysis. Emphasizes unifying principles. |

Figure 3: Institutional Course Description for MTE 483.

The majority of the curricula for the present study consisted of carefully sequenced problems for students to solve (in and out of the classroom setting) with an instructor guiding classroom discourse. The problems and students’ solutions were viewed as what Thompson (2002) refers to as didactic objects: they were the focus of discourse and served as the context in which mathematical conceptions and mathematical problem solving were discussed. The instructor of the sequence managed the discourse and solution presentations so that multiple solution paths were made visible. Students were encouraged to unpack the meaning of the central ideas (e.g., rate of change, linear growth) for solving both routine and non-routine problems. As the instructor for all
enactments of the experiment, I was able to make use of previous implementations as necessary in iterations of this project.

Design-based research methodology (Lesh & Kelley, 2002) was used to test and refine both the instructional sequence and the hypothetical learning trajectory. As discussed in Chapter Three, a design experiment typically refers to iterative refinement of some instructional innovation. The research design provided the means to study an instructional innovation, students’ interaction with the tasks used, and any growth or change in their mathematical behavior as a result of the instruction. Informed by the Multidimensional Problem-Solving (MPS) Framework (Carlson & Bloom, 2005), the instructional sequence (which served as the instructional innovation for this study) included thought-revealing activities that provided information about the students’ problem solving development. Results from each of the previous implementations were used to improve succeeding iterations of the course. For example, scaffolding was developed and refined for Extended Analyses Tasks (Stanley & Ramen, 2007; Bloom, 2007) after each implementation to PSMT’s in matematizing and generalizing (Bloom, 2007).

Subjects
The subjects for this study, Amy and Ben, were two PSMTs enrolled in the class Mathematics in the Secondary School at a large public university during the spring semester of 2004.

The class in which the data was collected enrolled of 19 undergraduate and 3 graduate students. There were 12 female students and 8 male students. Of the students, 16 were undergraduate prospective high school mathematics teachers, 2 had bachelor's
degrees and were enrolled in a post-baccalaureate program to attain teacher certification, and 2 were working toward a Ph.D. in mathematics education. Of the undergraduate students, all but two were traditional students (that is, students that progressed from high school into college with no interruption), with the remaining two returning to school after an absence of four or more years. The two post-baccalaureate students were also returning to school but after an absence of ten or more years. All students were anticipating being student teachers within a year. The class was composed of 13 white students, 4 Hispanic, 1 Chinese American, 1 Nigerian, and 1 Albanian. Amy and Ben can be thought of as typical students in the class as (like the majority of students in the class) they were both white and had followed the traditional path of entering college immediately after graduating from high school.

In the room where the instruction took place, students were seated at round tables, one group per table, to facilitate group work and discussion. All classroom sessions were recorded using a with camera operator using single digital camcorder. During whole class work and discussion, the operator was to focus on the whole class or the person who was speaking. During small group work, the camera operator focused on small group activity. There were three tables in close proximity to the camera. Within the first two weeks of instruction, I selected one of those tables as the group the camera would focus on during group work. This group was chosen because the members tested at different ability levels on their pre-instruction measures and they freely discussed the problems they worked on without prompting. Although other students rotated through the group, Ben and Amy consistently sat at that table. Ben and Amy were singled out for closer study because they represented the extremes of abilities in the class based on pre-semester measures.
Sources of Data

A design experiment “typically triangulates multiple sources and kinds of data to connect intended and unintended outcomes to processes of enactment” (The Design-Based Research Collective, 2003, p. 7). Therefore, this study drew from both qualitative and quantitative data sources in an effort to capture and validate changes in the attitudes, behaviors, and knowledge base of the PSMTs. Table 1 provides an overview of data sources. In the following section, I offer a more detailed description of the various sources of data and how the data was obtained and archived.
<table>
<thead>
<tr>
<th>Data Source</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAEP&lt;sup&gt;1&lt;/sup&gt; Instrument</td>
<td>To assess beginning knowledge base, confirm proficiency in high school math topics, and track change in knowledge.</td>
</tr>
<tr>
<td>VAMS&lt;sup&gt;2&lt;/sup&gt;</td>
<td>To gain insight into the subjects’ beliefs about mathematics and to capture changes in those beliefs.</td>
</tr>
<tr>
<td>Interviews – General questions</td>
<td>To gain insight into the subjects’ beliefs and attitudes about the value of competence, persistence, and conceptual knowledge and to track changes in those beliefs and attitudes.</td>
</tr>
<tr>
<td>Interviews – Problem solving sessions</td>
<td>To capture the problem solving behaviors of the subjects at the beginning and end of the instruction and to document changes in those behaviors.</td>
</tr>
<tr>
<td>Digital recordings of classroom sessions</td>
<td>To document classroom discussions in order to capture shifts in the subjects’ beliefs and attitudes regarding mathematics and mathematics teaching as well as their problem solving abilities.</td>
</tr>
</tbody>
</table>

---

<sup>1</sup> National Assessment of Educational Progress

<sup>2</sup> Views About Mathematics Survey
<table>
<thead>
<tr>
<th>Data Source</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Digital recordings of focus group</td>
<td>To document group problem solving sessions in order to capture students’ mathematizing, extending, and generalizing as they attempted to solve mathematical problems.</td>
</tr>
<tr>
<td>Problem sets</td>
<td>To gain insight into the resources that the subjects’ accessed to solve problems as well as identify areas where conceptual understanding is weak or has improved.</td>
</tr>
<tr>
<td>Reflective Journals</td>
<td>To assess student beliefs and attitudes regarding particular classroom activities, and triangulate with other data sources.</td>
</tr>
<tr>
<td>Class work</td>
<td>To gain insight into the resources that the subjects’ accessed to solve problems as well as identify areas where conceptual understanding was weak or has improved.</td>
</tr>
<tr>
<td>Extended Analyses</td>
<td>To gain insights into the subjects problem solving abilities and to document students’ efforts to generalize and extend mathematical situations on their own.</td>
</tr>
<tr>
<td>Instructor’s reflective journal</td>
<td>To gain insight into classroom activities from the teacher’s/researcher’s point of view</td>
</tr>
</tbody>
</table>
Research instruments

The present study made use of two published research instruments. These instruments were administered as both pre- and post-instruction measures.

The first instrument was a collection of 30 items on released versions of the 2003 grade 12 National Assessment of Educational Progress (NAEP) (National Center for Educational Statistics, 2001). A description of the NAEP mathematics assessment follows.

The NAEP mathematics assessment is based on the mathematics framework which describes the goals of the assessment. The framework provides

- The theoretical basis for the assessment, and
- The directions for what kinds of exercises should be included in the assessment, how those exercises should be designed, and how student responses should be scored.

The framework was the result of a comprehensive national process under the auspices of the National Assessment Governing Board. The framework is a broadly accepted outline of what hundreds of educators, curriculum experts, policymakers, and members of the general public thought the assessment should test.

After the completion of the framework, the NAEP Mathematics Committee worked with measurement specialists to create the assessment questions and scoring criteria according to the framework's specifications. All exercises and scoring criteria were carefully reviewed to ensure that the assessment met the requirements of the mathematics framework. (National Center for Educational Statistics, 2001a)
I specifically chose the NAEP items to assess students’ proficiency levels with content areas of high school mathematics because these items were created and scored using a specific framework and were used as a national mathematics assessment tool.

I selected 30 multiple choice items from the 2003 Grade 12 assessment. The selected items represented all content areas (Algebra and Functions, Number Sense, Data Analysis, Geometry, Measurement) and ability types (Problem Solving, Procedural Knowledge, Conceptual Understanding) which had a difficulty rating of hard. (Definitions for the five content areas and three ability types in mathematics can be found in Appendix A.) Figure 4 provides an example of a NAEP item used on the pretest and posttest which was classified as assessing knowledge in the content area identified as Measurement and ability type identified as Problem Solving.

6. A rectangular pool 24 feet long, 8 feet wide, and 4 feet deep is filled with water. Water is leaking from the pool at the rate of 0.4 cubic foot per minute. At this rate, how many hours will it take for the water level to drop 1 foot?

A) 4          B) 8          C) 12          D) 16          E) 32

Figure 4: Problem #6 from the pre and posttest.

The tables below show the characterization of the instrument items by type of ability (Table 2) and content area (Table 3). The complete pre and posttest is provided in Appendix A.
Table 2

*Categories of Instrument Items According to Ability Type*

<table>
<thead>
<tr>
<th>Type of Ability</th>
<th>Number of Items</th>
<th>Percent of Instrument</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem solving</td>
<td>13</td>
<td>43%</td>
</tr>
<tr>
<td>Procedural knowledge</td>
<td>7</td>
<td>23%</td>
</tr>
<tr>
<td>Conceptual understanding</td>
<td>10</td>
<td>33%</td>
</tr>
</tbody>
</table>

Table 3

*Categories of Instrument Items According to Content Area*

<table>
<thead>
<tr>
<th>Content Area</th>
<th>Number of Items</th>
<th>Percent of Instrument</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra and Functions</td>
<td>10</td>
<td>33%</td>
</tr>
<tr>
<td>Number Sense</td>
<td>8</td>
<td>27%</td>
</tr>
<tr>
<td>Data analysis</td>
<td>4</td>
<td>13%</td>
</tr>
<tr>
<td>Geometry</td>
<td>4</td>
<td>13%</td>
</tr>
<tr>
<td>Measurement</td>
<td>4</td>
<td>13%</td>
</tr>
</tbody>
</table>

The second research instrument administered was the Views About Mathematics Survey (VAMS) (Carlson et al., 1998; Carlson, 1999a). VAMS was designed to identify differences between the views of students and mathematicians, identify patterns in student views, classify sets of views in general profiles, and measure how effective instruction is at changing student views.

The survey comprised 30 statements about various aspects of mathematics and mathematics; each of the 30 statements was followed by two contrasting statements, $a$ and $b$, which the students were asked to consider, and then select a number from one to
seven, representing whether they believe \( a \) or \( b \) is more true, each is partially true, or each is equally true. Figure 5 provides an example of how the VAMS items were structured. Student responses could then be evaluated according to specific criteria to determine if they held \textit{expert}, \textit{upper transitional}, \textit{lower transitional}, or \textit{naïve} views about the practice, teaching, and learning of mathematics (Carlson et al., 1998).

In the context of the present study, VAMS was used to both assess the beliefs of the students at the beginning of the class as well as to measure any change that occurred as a result of the intervention. Additionally, results from VAMS were triangulated with behavior observed during the interviews, in the classroom and views expressed in reflective journals. The complete VAMS instrument is included in Appendix B.

In summary, two research instruments were employed to assess students before and after instruction. The NAEP pre and posttest was used to assess mathematical competence, and VAMS was used to assess mathematical beliefs and attitudes. Together these two sources of data provided the quantitative measures for the study. The following sections discuss the sources of qualitative data.
Learning mathematics requires:

(a) a serious effort.
(b) a special talent.

What would each one of the seven choices mean?

<table>
<thead>
<tr>
<th></th>
<th>Towards “Only (a)”</th>
<th>Equally (a) &amp; (b)</th>
<th>Towards “Only (b)”</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Only (a), Never (b): Learning mathematics requires <em>only</em> a serious effort and <em>no</em> special talent <em>at all.</em></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Mostly (a), Rarely (b): Learning mathematics requires <em>far more</em> a serious effort than a special talent.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>More (a) Than (b): Learning mathematics requires <em>somewhat more</em> a serious effort than a special talent.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Equally (a) &amp; (b): Learning mathematics <em>equally</em> requires <em>both</em> a serious effort and a special talent.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>More (b) Than (a): Learning mathematics requires <em>somewhat more</em> a special talent than a serious effort.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Mostly (b), Rarely (a): Learning mathematics requires <em>far more</em> a special talent than a serious effort.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Only (b), Never (a): Learning mathematics requires <em>only</em> a special talent and <em>no</em> serious effort <em>at all.</em></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Figure 5:* Example of a contrasting alternatives item from VAMS.

*Interviews*

Amy and Ben participated in two semi-structured task-based interviews (Goldin, 2000) conducted by me (instructor and researcher). Both the pre and post instruction interviews lasted about 60 minutes and were videotaped by a single unmanned camera. The interviews had two distinct components. The first component consisted of general questions (Figure 6) designed to glean information about Ben and Amy’s mathematical
and pedagogical experiences prior to participation in the experimental class. The second part was a problem solving session designed to provide information about their problem solving abilities.

<table>
<thead>
<tr>
<th>Pre-Instruction Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General Questions</strong></td>
</tr>
<tr>
<td>1. What part of math are you really good at? (as in calc., alg., geometry, etc.)</td>
</tr>
<tr>
<td>2. What do you think of as problem solving?</td>
</tr>
<tr>
<td>3. How would you describe your problem solving abilities?</td>
</tr>
<tr>
<td>4. What actions, behaviors, and beliefs most influence your success in solving problems?</td>
</tr>
<tr>
<td>5. What experiences have contributed most to the development of your problem solving ability?</td>
</tr>
<tr>
<td>6. How do you prioritize actions you take during problem solving?</td>
</tr>
<tr>
<td>7. How do you manage frustration? Do you tend to persist or give up?</td>
</tr>
<tr>
<td>8. How confident are you in your knowledge of high school mathematics?</td>
</tr>
<tr>
<td>9. What high school math class do you most look forward to teaching and why?</td>
</tr>
</tbody>
</table>

*Figure 6: General questions used for the pre-instruction interview.*

Amy and Ben were also presented with three non-routine problems (Figure 7). As the subjects attempted to solve the problems, they were prompted to describe their thinking aloud as they were solving the problems. These problems were selected because, although they required only the use of foundational mathematical concepts, they were
sufficiently novel and complex as to elicit dead ends and emotional responses. In addition, the same problems, when presented to mathematicians in the Carlson and Bloom (2005) study, provoked a rich variety of responses and solution paths.

<table>
<thead>
<tr>
<th>Pre-Instruction Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem Solving Session</td>
</tr>
</tbody>
</table>

1.) A square piece of paper ABCD is white on the front side and black on the back side and has an area of 3 square inches. Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is $\frac{1}{2}$ white and $\frac{1}{2}$ black. How far is A' from the fold line?

2.) List all the integers less than 100 that are divisible by 2, 3, 4, 5, and 6.

3.) Two numbers are "mirrors" if one can be obtained by reversing the order of the digits (i.e., 123 & 321 are mirrors). Can you find:

   (a) two mirrors whose product is 92565

   (b) two mirrors whose sum is 8768

   **Answers:** (1) 1, (2) 60, (3a) 165 and 561, (3b) no

*Figure 7:* Mathematical problems used for the pre-instruction interview.

Amy and Ben were also interviewed again after the completion of the instructional sequence. These interviews also used general questions in conjunction with a problem solving session. The general questions (Figure 8) were designed to probe the subjects’ perception of the impact of the course on his or her mathematical development.
Post-Instruction Interview

General Questions

1. How would you say that your knowledge base has grown/changed this semester? Can you characterize it in some way? Like deeper, more connected, richer…can you give an example?

2. How would you say your problem solving abilities have changed this semester? Can you offer examples?

3. What activities were especially influential or revealing?

4. Do you think this class has influenced your work in other mathematics classes?

5. Do you think you have more confidence in the High School math topics than you did before?

6. How do you think you benefited from the extended analyses?

Figure 8: General questions used for the post-instruction interview.

As in their first interview, Amy and Ben were presented with three non-routine mathematics problems to solve (see Figure 9). As before, all three were chosen for their ability to elicit dead ends and strong emotions.
Post-Instruction Interview

Problem Solving Session

1. Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that is in the bottle.

2. The quarter circle shown above has center C and radius 10. If the perimeter of rectangle CPQR is 26, what is the perimeter of the shaded region?

3. Of all lines through the point (5,2), which one cuts off the region of smallest area in the first quadrant?

Answers: (1) (2) $5\pi + 17$ (3) $y = \frac{2}{5}x + 4$.

Figure 9: Mathematical problems from second interview.

Digital recordings of classroom sessions

All classroom sessions were recorded by a camera operator using a single camcorder. During whole class work and discussion, the operator was instructed to focus on the whole class or the person who was speaking. Previous iterations of the design
experiment revealed that discussion of homework yielded rich data about PSMTs’ understanding of particular concepts and content areas and highlighted areas where students’ conceptual understanding was weak or relied primarily on procedures (Bloom, 2004b).

During small group work, the camera operator focused primarily on the group which included Amy and Ben. Recording small group interaction during group problem solving sessions was intended to provide documentation of the subjects’ mathematical behaviors in a more social and natural environment than that of clinical interviews. Focusing on Amy and Ben’s group throughout the semester was intended to provide longitudinal evidence of changes as the group and their practices evolved.

*Written work*

All written work submitted by the students during the experiment was archived. I photocopied all the assigned problem sets before they were graded to produce clean artifacts. Reflective journals, and extended analysis tasks were also copied and archived for retrospective analysis. A detailed description of these data is provided in Chapter Five.

*Instructor’s reflections and research field notes*

As the instructor/researcher, I maintained an audio journal as well as field notes of my thoughts and impressions as the experiment progressed.

*Summary of the Research Design*

The present study is the fifth iteration of a design experiment. The study was situated in a mathematics course required by prospective high school mathematics teachers. Amy and Ben were selected as subjects for case studies.
Data sources included quantitative measures, administered before and after the enactment of the instructional sequence, as well as qualitative data, collected before, during, and after the enactment of the instructional sequence. The variety of data collected during this project provided me with the means to conduct a detailed retrospective analysis of the mathematical development of the subjects.

Methods of Data Analysis

An overview of the methods used for analyzing the data collected is provided in this section. This is followed by a description of the methods used for each of my data sources.

Case Study Methods

The goal of the present study was to understand and explain the problem solving behavior of PSMTs as they engaged in the instructional sequence. According to Yin (2006), case study methods are appropriate when the research goal is to address the evolution of behaviors over time as well as account for why changes occurred. For this reason, I decided to make case studies of two students for the ability of case studies to focus on development over time. The wide variety of data collected for this study provided a broad data set for explaining PSMTs’ behaviors throughout the semester (Stake, 1998) and comparing and contrasting two cases (Yin, 2006) enriches the results of and conclusions drawn from this study.

Amy and Ben were selected as the subjects for the case studies because:

- They represented typical PSMT’s in the class
- Their pre-instruction measures revealed they had different mathematical backgrounds and differing views about mathematics
• They routinely sat together, and freely shared their thinking with little prompting

Other members of the group were not offered for case studies because I felt their stories would not be significantly different from those of Ben and Amy

Amy represents the case of a PSMT whose pretest scores were below the average for the class. A detailed study of Amy describes the evolution of a student with mathematical and problem solving skills that were not particularly strong at the onset of the experiment. Ben, on the other hand, represents the case of a student whose pretest scores were among the highest in the class. A detailed study of Ben describes the evolution of a student whose mathematical skills and problem solving behaviors appeared strong at the beginning of the experiment.

*Pre and Posttest*

The pre and posttests for all students enrolled in the class were scored and recorded. Both scores for each student in the class were then compared to looks for trends revealed by the scores. Amy’s and Ben’s scores were singled out for closer analysis. Their individual pre and posttests were examined for insights into their content knowledge of mathematics prior to the instructional sequence, and at the completion of the experiment. Their scores were also compared to the other students enrolled in the class.

*VAMS*

Student responses to 22 of the 30 VAMS items were recorded and categorized as expressing *expert, mixed, or folk views* according to the guidelines provided by Carlson, et al. (1998). According to the authors, *expert views* are those aligned with the views of
mathematicians, folk views are those views that were opposite the expert views, and 
mixed views are those in between. The remaining 8 of the 30 items, which did not have 
associated response categories, were also recorded since the items still offered valuable 
information about the beliefs and attitudes of the survey taker. Once the responses were 
classified, the number of responses in each of the three categories was counted according 
to a prescribed scheme (Carlson et al., 1998) to determine if the survey indicated 
someone espousing expert, upper transitional, lower transitional or naïve views about 
the practice, teaching, and learning of mathematics. Once the pre-instruction and post-
instruction surveys were analyzed, PSMTs’ placement in these categories for both 
surveys were compared to determine if there were shifts in the number of PSMTs in the 
categories of expert, upper transitional, lower transitional, or naïve. Amy’s and Ben’s 
individual responses were also examined for information about their mathematical 
attitudes and beliefs. These results were also compared to other data sources (e.g., pre and 
post-instruction interviews, written reflections), to search for inconsistencies or 
corroboration between self-reported and observed behavior.

Pre and Post-Instruction Interviews

Written artifacts from Amy’s and Ben’s problem solving sessions were 
photocopied and the recordings of both the general questions and problem solving 
components of the interviews were transcribed.

In the first pass through the pre-instruction interviews, responses to the general 
questions were analyzed through the lens of the Multidimensional (MPS) Framework 
(Carlson & Bloom, 2005). Amy’s and Ben’s statements regarding the phases of problem 
solving were noted, as were any mention of confidence, persistence, or other of the
affective components of problem solving. Next, these statements were compared to the results of the pretest and VAMS for confirming or disconfirming evidence. Finally, Amy’s and Ben’s statements were compared to each other for areas of similarity and difference.

Their responses to the general questions from the post-instruction interview were handled in much the same way. Of particular interest in this second round of questions were the PSMTs’ self-reports of changes in their knowledge base and problem solving abilities and their attributions for those changes.

The problem solving sessions that took place as part of both the pre and post-instruction interviews were transcribed. The problem solving sessions were coded using the MPS Framework. Pre- and post-instruction data were then compared examine any changes in problem solving behaviors.

**Digital Recordings of Classroom Sessions**

Recordings of each class session were viewed as videos were uploaded immediately following each class. Although the formal analysis did not take place until later, adjustments to the next instructional session or in instructions to the camera operator were sometimes made after these initial informal reviews.

For the first formal pass through the video data, I created researcher narratives of each class session. These researcher narratives contained detailed descriptions of the class activity and of behaviors or statements made by particular students that appeared to be important. Based on a retrospective analysis of the narratives, I divided the semester into three segments for the purpose of organizing and reporting of the data. The first segment consisted of the first five class sessions. The second segment started at the sixth
class session and ended with the fifteenth session. The third segment began when students returned from the break and lasted to the end of the instructional sequence. Shifts in Amy’s and Ben’s problem solving behaviors were characterized as they moved from one segment of the semester to another.

After a second retrospective review, selected class sessions were sent for formal transcription. Videos were viewed again, this time with the transcripts and the researcher narratives as a guide. Finally, the recordings were viewed numerous times with particular focus on the actions and utterances of Amy and Ben.

Written Work

All written work was read and studied for general trends in problem solving and problem solving behaviors in the class as a whole. Ben and Amy’s work was then separated for fine grained study.

Amy’s and Ben’s submitted problem sets also were studied for evidence of conceptual understanding. Errors were scrutinized for insights into the mathematical understandings and misunderstandings they revealed. The work was then studied chronologically for evidence of growth and change. Finally, insights were compared and correlated with findings from the video data.

Extended Analysis Tasks submitted by Amy and Ben were analyzed individually for evidence of abstraction, generalization, and mathematization as described by Stanley and Sundstrom (2007). Then they too were studied chronologically for evidence of growth and change. As with the problem sets, results from this analysis were compared and correlated with the video data.
Reflective journals were studied for insights into student thinking and beliefs as well as for triangulation with other data sources.

*Longitudinal and Retrospective Analyses*

Once the data for Amy was analyzed as described in the previous sections, all the data was separated into five units: pre-instruction (the pretest, VAM and the pre-instruction interview), beginning of the semester (sessions 1-5), middle of the semester (sessions 6-15), end of the semester (sessions 16-22), and post instruction (the posttest, VAMS and post-instruction interview). The respective data for each segment were compared for trends and inconsistencies, resulting in a story documenting Amy’s evolution of mathematical and problems solving behaviors. Ben’s data was handled in the same manner, resulting in his story. In both cases, the data were analyzed both retrospectively and longitudinally.

*Summary of Methods*

In this chapter, I have described the research design for this investigation. I have provided descriptions of the case study subjects (Amy and Ben) and the setting for the study. Sources of quantitative and qualitative data have been provided, as well as descriptions of the means of analysis of that data. Finally, I have described longitudinal and retrospective analyses that resulted in the case studies in found in Chapter Six and Chapter Seven.
CHAPTER FIVE

THE INSTRUCTIONAL DESIGN

In this chapter, I present a detailed discussion of the instructional design of my study. I explain my purpose and lay out my justifications for making certain research decisions relative to the goals of the instructional sequence; I also discuss my premises that underpin the design. By the end of this chapter, I intend for the reader to have a clear idea of the nature of the tasks used in the teaching experiment and the principles that guided my selection of them.

Instructional Goals

In designing the instructional sequence for my study, I was motivated by national calls, both from the National Council of Teachers of Mathematics (NCTM) and the Conference Board of Mathematical Sciences (CBMS), for the improvement of teacher preparation in mathematics. Both organizations have emphasized the need for flexible, coherent understandings of the mathematics concepts and content at all grade levels (NCTM, 2000; CBMS, 2000). In particular, CBMS recommends a capstone course for prospective secondary mathematics teachers (PSMTs) that promotes deep understanding of the content of high school mathematics and its connections to the more abstract mathematics PSMTs study in their undergraduate program. In response to these calls and specific recommendations below from NCTM (1991), the overarching goals of the instructional sequence I designed for PSMTs were to facilitate movement in the following directions:

- toward classrooms as mathematical communities - away from classrooms as simply a collection of individuals;
• toward logic and mathematical evidence as verification - away from the teacher as the sole authority for right answers;
• toward mathematical reasoning - away from merely memorizing procedures;
• toward conjecturing, inventing, and problem solving - away from an emphasis on mechanistic answer-finding;
• toward connecting mathematics, its ideas, and its applications - away from treating mathematics as a body of isolated concepts and procedures (p. 3).

Based on the findings of the Carlson and Bloom (2005) study described in Chapter Three, I conjectured that a course based in the activity of mathematical problem solving and where problems and solutions would act as what Thompson (2002) refers to as didactic objects could promote such movement. I used the Multidimensional Problem-Solving (MPS) Framework as a guide for designing tasks and activities in the instructional sequence.

The MPS Framework provided benchmarks for mathematical behaviors and practices the instruction would purposefully promote, as well as the ways of thinking and reasoning that experienced problem solvers routinely employed. Through the activity of problem solving, prospective secondary mathematics teachers (PSMTs) would learn to orient themselves to a problem, conjecture about solution paths, plan, execute, and check as they solved problems. Desired learning outcomes of the instructional sequence also included evaluating mathematical resources, managing frustration, and the justification
and verification of claims and conjectures. In this way, the MPS Framework guided the
task creation and implementation.

In summary, these were the instructional goals for the design. Later in this
chapter, I discuss specific tasks in the instructional sequence in terms of how they relate
to these goals.

Guiding Premises

As I was making research decisions at each step in designing the instructional
sequence, I held certain premises that guided me. These included the following:

1. The attitudes and behaviors observed in mathematicians as they solved
   problems are desirable and consistent with the habits of mind needed for
teaching mathematics.

2. If problem solving is at the heart of mathematics, as some claim,
   mathematics teachers should be skilled and experienced problem solvers.

3. A compartmentalized conception of mathematics does not support
   attempts to teach mathematics as a coherent and growing body of
   knowledge.

4. PSMTs need immersion in the language of mathematics. That is, they
   require experience in:
   - Following someone else's reasoning critically
   - Explaining their own thinking in multiple ways
   - Devising appropriate representations to explain their thinking
   - Writing those representations and explanations for others
5. PSMTs rarely think of themselves as mathematicians or of being capable of making mathematical discoveries; moreover, they do not regard this as being important for teaching mathematics.

6. It is difficult to enact teaching strategies that PSMTs have not witnessed or experienced.

These premises – some grounded in the literature, others grounded in the wisdom of practice – guided the creation of the instructional design. Next, I provide some background on each of my premises and, in some cases, elaborate on how they guided the development of the learning trajectories and mathematical tasks.

**Premise 1**

*The attitudes and behaviors observed in mathematicians as they solved problems are desirable and consistent with the habits of mind needed for teaching mathematics.*

The problem solving behaviors of mathematicians observed in the Carlson and Bloom (2005) study that are also seen as useful for mathematics teachers include conjecturing, curiosity, mathematical integrity, and a coherent conceptual understanding of mathematics. I conjectured that problem solving should be an appropriate task to foster these behaviors. Solving problems alone, however, would be insufficient to foster these ways of thinking. Mathematical integrity, curiosity, and the ability to conjecture from a well-developed conceptual base could be enhanced not only by solving problems but also by sharing solutions and reasoning about solutions in a public way. Problems solved individually could be presented by several PSMTs using different strategies and different ways of thinking.
In my teaching experience, I found that it was instructive for students to see that there was rarely only one way to solve a problem. In my work with preservice elementary teachers, I found that it was typically difficult for them to imagine that others thought differently than they did. To foster their development, I would design activities that emphasized the idea of multiple solution paths to help them consider other ways of understanding mathematics. It is important to note that the success of these activities was contingent on the instructor using the multiple solutions approach as a vehicle for unpacking key ideas and connecting topics, concepts, and procedures for the purpose of nurturing more coherent and conceptual understandings of school mathematics.

**Premise 2**

*If problem solving is at the heart of mathematics, as some claim, mathematics teachers should be skilled and experienced problem solvers.*

When asked how they became good problem solvers, many of the mathematicians in our study said they had had a lot of practice (Carlson & Bloom, 2005). Reflecting on this, I recognized that in my own education, "story problems" were relegated to the end of the chapter, and they were easily skipped if the teacher ran out of time. As a new teacher, I was often frustrated by my students' refusals to even attempt what I considered the fun problems. As a result, I undertook the challenge to devise strategies to engage my students in the problem solving process. I found some successful strategies including assigning more problems, allowing the students to solve problems in groups, and starting new topics with an interesting problem. I also came to realize that my own problem solving abilities influenced the tasks I selected and how I assessed my students’ work. In short, just as a lot of practice in problem solving was characteristic of the expert
mathematicians in the Carlson and Bloom study, I knew that “a lot of practice” must also be an important component of teacher education.

In creating this project, I envisioned a course where problem solving was the vehicle for all mathematical discussion. In other words, the problems would generate the curriculum. I conjectured that if PSMTs practiced problem solving, they would become more comfortable with problem solving tasks and more confident with their abilities. If they were more confident and comfortable with problem solving, they might be more likely to include real problem solving tasks in their own classrooms.

**Premise 3**

*A compartmentalized conception of mathematics (i.e., as individual concepts and procedures) will hinder attempts to teach mathematics as a coherent and growing body of knowledge.*

Currently in the United States, according to Stigler and Heibert (1999), most mathematics curricula promote the notion that mathematics is a set of facts and procedures to be mastered. Typically, the high school curriculum is designed to allow a student to progress in sequence through Algebra I, Geometry, Algebra II, Pre-Calculus, and Calculus. (Some schools have an alternate “integrated” approach, but even that tends to chunk together topics.) However, these course delineations reflect quite arbitrary separations of topics that do not exist in many places outside of the United States.

To facilitate the movement toward connecting mathematical ideas, procedures, and applications, I created problem sets in the present study that drew from different areas of mathematics. For instance, Problem Set 9 (Figure 10) contains problems that
require the use of multiple concepts such as exponential growth/decay, combinations, systems of equations, trigonometric ratios, and geometry.

To illustrate how such a problem set could promote making connections between different mathematics topics, consider problem #2 from the set. There are two ways one could look at this problem. One could look at it as the series

$$\sum_{n=1}^{8} n = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36.$$  
A very different point of view would be to see it as the combination of nine things taken two at a time, or $$C_2 = \frac{9!}{(9-2)!2!} = 36.$$  
Without concern for the context of a particular section in a textbook at hand, one can acknowledge that both perspectives are valid as well as demonstrate that both strategies produce equivalent results.

Considering problem #6 from the same problem set, we see that one way this problem can be solved is by placing the figure in the Cartesian plane and using properties of linear functions. This solution approach allows the student to connect geometry and algebra. My hypothesis was that as PSMTs are required to scan their knowledge base (rather than the preceding section in their textbook) to solve problems and then share those solutions in class, they will learn to make connections in mathematics and develop a habit of exploring alternative solution approaches to the same problem.
Problem Set 9

1. The rate of decay of a radioactive substance is proportional to the amount of substance present at any time \( t \). In 1840 there were 50 grams of the substance and in 1910 there were 35 grams. To the nearest gram, how many grams of the substance remain in 1990?

2. How many connecting cables are needed in order that any two of nine offices in a building can communicate directly?

3. Generalize #2. That is, find how many cables are needed if there are \( n \) offices.

4. A small corporation borrowed $500,000 to expand its product line. Some of the money was borrowed at 9%, some at 10%, and some at 12%. How much was borrowed at each rate if the annual interest was $52,000 and the amount borrowed at 10% was two and a half times the amount borrowed at 9%.

5. A builder wishes to construct a ramp that is 24 feet long, and which rises to a height of 5 feet above the level ground. Approximate the angle that the ramp should make with the level ground.

6. In the adjoining figure, \( AB \) and \( BC \) are adjacent sides of square \( ABCD \); \( M \) is the midpoint of \( AB \); \( N \) is the midpoint of \( BC \); \( AN \) and \( CM \) intersect at \( O \). Find the ratio of the area of \( AOCD \) to the area of \( ABCD \).

\[ \text{Figure 10: Problem Set 9.} \]
**Premise 4**

*PSMTs need immersion in the language of mathematics. That is, they require experience in:*

- Following someone else's reasoning critically
- Explaining their own thinking in multiple ways
- Devising appropriate representations to explain their thinking
- Writing those representations and explanations for others

To create the types of classroom communities that truly engage in rich mathematical discourse as endorsed by the National Council of Teachers of Mathematics, among others, teachers need to be skilled in following a student’s reasoning while at the same time assessing the validity of that reasoning. Using a variety of representations, they also need to explain concepts in multiple ways. As a novice teacher, my skills were underdeveloped in both areas. I found that it was very difficult to track someone else's reasoning, especially when different from my own. Additionally, when I compared the problem solving behaviors of mathematicians and PSMTs, I noticed that the mathematicians were much more articulate when explaining their solutions than the undergraduates I was interviewing (Bloom, 2002). I hypothesized that the PSMTs lacked both the experience and the language needed to express themselves clearly.

In the instructional design of the present study, the instructor explicitly requires all students to share their reasoning and to listen and respond to others’ explanations. PSMTs are responsible for assessing the validity of an argument or solution path; the instructor rarely endorses a particular solution.
**Premise 5**

*PSMTs rarely think of themselves as mathematicians or of being capable of making mathematical discoveries; moreover, they do not regard this as being important for teaching mathematics.*

When I finished my undergraduate degree in mathematics, I still did not know what mathematicians did. The only hint I had was from my Discrete Mathematics professor, Michelle Wachs Galloway, who said that mathematicians create new mathematics. At that time, I thought that that work was far beyond my abilities. It was only when my own mathematical explorations led to surprising (for me) discoveries that I started to see that I could do the work of a mathematician. NCTM calls for teachers to help students explore, conjecture, and make discoveries. In the tradition of Realistic Mathematics Education (RME), Gravemeijer (2000) promotes the concept of “guided re-invention” as a vehicle for immersing students in mathematical activity. That is, students should be offered opportunities to invent or create fundamental concepts in mathematics. Accordingly, emphasis in the classroom should therefore be placed on *mathematizing* activities rather than memorizing algorithms and procedures. In this project, I have presumed that teachers need to have the experience of discovering and creating mathematics for themselves. For this reason, I included Extended Analysis Tasks (described later in this section) because these proved to be effective tools for promoting mathematical thinking and discovery in my previous study (Bloom, 2007).
Premise 6

It is difficult to enact teaching strategies that PSMTs have not witnessed or experienced.

PSMTs who have matriculated through the traditional mathematics curriculum in the United States generally share the same experience of learning via the typical model for teaching and learning mathematics (Stigler & Hiebert, 1999). Their experience has most likely been the following: first, the teacher goes over and collects homework; then there is a short explanation of that day’s topic; next, the teacher works several examples of the upcoming assignment for the students and then assigns seatwork so the teacher can help individual students (group work and technology may also be included). This is the general routine in many high school and college mathematics classrooms.

As reported in *The Teaching Gap* (Stigler & Hiebert, 1999), this teaching routine is based more on culture than on what is known about the teaching and learning of mathematics. Therefore, in the present innovation, the instructor must implement teaching strategies that have been found effective in research on the teaching and learning of mathematics. In particular, the instruction must be student centered, thus allowing PSMTs to be directly engaged in mathematical tasks and discourse.

At this point, I turn from my discussion about the underlying premises that guided the instructional design to an explication of the different types of tasks that made up the instructional sequence.

Selection of Tasks

To achieve my research goals, I developed (and borrowed) and sequenced tasks I believed would foster in PSMTs problem solving behaviors and dispositions as well as coherent mathematical conceptions. At the same time, I wanted to address affective
aspects of learning such as confidence, integrity, curiosity and self efficacy. I employed three different types of tasks: Homework Problem Sets, Problems that were solved during class, and Extended Analysis Tasks. In this section, I describe examples of each type of task in detail, and I provide my rational for selecting them as part of the instructional sequence.

*Homework Problem Sets*

I created and refined 18 problem sets for PSMTs to work on independently as homework; I actually used 17 of them (Appendix C) in the instructional sequence. As described in the preceding section, each problem set contained problems that encompassed a variety of mathematical topics and skill levels. Although the problems may seem random, they were purposefully produced.

Almost every set contained at least one problem that was included for its ability to uncover mathematical misconceptions or assumptions and thus generate mathematical discourse. The Highway Fencing Problem (Figure 11) from Problem Set 4 is an example of such a problem.

A certain farm allots $1000 for fencing a rectangular area that is to abut a highway. Because the fencing on the highway side must be attractive, it costs $4 per lineal foot. The other three sides of the area are fenced at $2 per foot. What are the dimensions of the rectangle that maximizes its area?

*Figure 11:* The Highway Fencing Problem

During my years of teaching, I have often observed that it is not unusual for some students to first assume the rectangle in question must be a square and base their solution on that initial error; other students, however, will not make that assumption. When both solutions are presented, the instructor can use the different results to guide the class
toward questioning whether the assumption of “square-ness” was appropriate in the face of other constraints included in the problem.

Most sets included a symbol manipulation problem like this problem from the same set:

\[
\text{Prove that } \cot^2 x = \frac{1 - \sin^2 x}{\sin^2 x}
\]

*Figure 12:* An example of a symbol manipulation problem.

Solving problems without the usual cues that many students rely on can be difficult and disheartening. I conjectured that including a problem they can be successful with would bolster their confidence.

Sometimes problems are included to prepare students for future tasks. For instance, this problem from Problem Set 13, “Select any odd number. Square it and subtract 1. Prove that the result will always be divisible by 8,” sets up the more difficult proof in set #14, “Show that the product of three consecutive integers, the smallest which is even, is divisible by 24.”

For the first six problem sets, students were required to list any mathematical concepts that were used to solve the problem. The point of this exercise was to force the PSMTs to reflect on the mathematics they have just used. Previous iterations of the design experiment showed that this exercise facilitated the development of metacognition while it also helped them to think critically about the skills and understandings a student might need to solve the problem (Bloom, 2004). The question, "Where might a problem like this be found in the curriculum?" was often asked during discussions of homework
solutions. Students were also required to list any problem solving strategies or *heuristics* they made use of. Again, the purpose was to have PSMTs reflect on tools they might use automatically and make a conscious note of their problem solving strategies.

*Problems Solved During Class*

Most class sessions centered on solving challenging problems in groups and presenting the solutions to the class. Appendix D provides a complete listing of the problems used.

Some of the class problems were chosen for their ability to highlight a specific heuristic. For instance, the Magic Square Problem (Figure 13) was used to draw attention to the strategy of setting sub-goals. To solve this problem, one must first identify the number in the center of the square and the common sum for the rows and columns, making these logical sub-goals in the solution process.

In your group, create a 3x3 “Magic Square” that uses the numbers 1, 2, 3, 4, 5, 6, 7, 8, & 9. Be prepared to explain your process.

*Figure 13:* The Magic Square Problem.

Other problems were chosen because they commonly generated multiple solution paths. In the next section, I describe the evolution of the design of a particular sequence of activities based on one problem.

*Evolution of the Triangle Problem sequence*

In this section, I discuss the development of several weeks of classroom activity based on a problem used by Schoenfeld (1985a) in his problem solving course (Figure 14).
In the first iteration of the instructional sequence, I included the Triangle Problem as a classroom activity because it was challenging and because I anticipated that it would elicit multiple solution approaches.

The Triangle Problem

You are given a fixed triangle $T$ with base $B$ as below. Show that it is possible to construct, with a straightedge and compass, a straight line that is parallel to $B$ and that divides $T$ into two parts of equal area.

If you figured that one out, can you similarly break it into 5 parts of equal area?

*Figure 14:* The Triangle Problem.

As I observed that first group of students struggle with the problem, I noticed the following:

1. The solution of this problem requires the use of algebraic manipulation of proportional statements as well as geometric constructions, yet I noticed that students tended to use one construction to the exclusion of the other. That is, they tried to solve the problem by either making geometric constructions or defining relationships algebraically. Students who failed to use both approaches made little progress.
2. One group of students stopped working on the problem when they became frustrated. They waited for the instructor to tell them how to proceed rather than persist in their efforts.

3. Students were surprised that one problem was the focus of several days of classroom activity. The experience of such sustained effort on a single problem was new to them.

4. Some students either had little or no experience with compass-and-straight-edge constructions or had difficulty accessing the geometric theorems needed to make the construction.

Based on these observations, I made several modifications to the activity. In the next iteration of the course, I preceded the Triangle problem with another, easier, construction problem also found in Schoenfeld’s (1989a) work (Figure 15) and I provided some basic instruction on geometric constructions.

Inscribed Circle Problem

You are given two intersecting straight lines and a point $P$ marked on one of them as shown below. Show how to construct, using a straight edge and compass, a circle that is tangent through both lines and has the point $P$ as one of the points of tangency to one of the lines.

![Figure 15: The Inscribed Circle Problem](image-url)
Additionally, at the end of the first full class period that was spent solving the Triangle Problem, I asked the students to submit a written reflection on the role of frustration in their problem solving process, and to also describe under what circumstances frustration provided them with motivation and when frustration was not motivating. Carlson and Bloom (2005) found that the mathematicians they studied expressed frustration during their problem solving sessions, yet they appeared to have developed ways to manage their frustration so that it was productive (rather than destructive) in their efforts to solve problems. Therefore, my purpose for this assignment was to provide an opportunity for PSMT’s to examine and confront their frustration at a time when they were frustrated by the Triangle problem. The last modification I made was to ask each group to present their solutions to both of the assigned constructions during class.

These adjustments provoked frustration while also providing adequate support to the PSMT’s for coping with the frustration they were experiencing. The resulting written journals revealed that the students acknowledged that frustration was a natural part of the problem solving process. Group presentations revealed that while the students completed the constructions properly, they could not mathematically justify the steps in the construction. For example, a group might be able to locate a point on the opposite ray by drawing an arc with the compass and claim it was the same distance from the vertex as \( P \) in the Inscribed Circle Problem (Figure 15), but not be to provide a valid mathematical justification for that claim.

In order to support the development of mathematical argumentation in this activity, I again made several modifications before the next enactment of the instructional
sequence. I moved the Arrowhead Problem (Figure 16) so that it appeared in the problem
set completed immediately before the beginning of the Inscribed Circle problem activity.

The tip of an arrow is to be in the shape of the figure
bounded by a circle of radius ½ inch and two tangent lines
to this circle drawn from a point 1 inch from the points
of tangency. Find the area of arrow tip ABC.
(The region shaded in light gray)

Figure 16: The Arrowhead Problem

I intended to use the Arrowhead Problem to set up the upcoming geometric construction
problems, as it afforded the opportunity to discuss several theorems that are used in the
construction of the Inscribed Circle Problem. In particular, the theorem stating that the
radius of a circle is perpendicular to the tangent is useful in both problems. Additionally,
when the instructor interacted with the groups as they solved the problems, she
emphasized the need to be able to mathematically justify each step in the construction
when presenting the solution.

In addition, a reading assignment was included for the purpose of introducing
students to research literature on mathematical problem solving. Students were asked to
read the first chapter of Schoenfeld's book *Mathematical Problem Solving* (1985a). In
this chapter, Schoenfeld not only outlines his theories about problem solving, but he uses
transcripts of undergraduate students solving the Inscribed Circle Problem and the
Triangle Problem to illustrate his points. The subsequent class discussion about the
reading revealed that the PSMTs appreciated the information about problem solving.
They also revealed that, having solved the same problems prior to the reading, the transcripts of the problem solving sessions helped make the research relevant to them.

In the final adjustment to the sequence of activities generated by the Triangle Problem, students were asked to submit a written summary of the main ideas from the Schoenfeld paper, and to reflect upon their personal experience solving the Triangle Problem. In these written reflections, students appeared to confront long held beliefs about problem solving as well as identify their personal problem solving difficulties.

In this way, an interesting problem evolved into a sequence of activities that provided students with the experience of sustained effort on one problem, confronting and managing frustration, and the opportunity to persist in their efforts to solve the problem. Additionally, the activity of reading the first chapter from Schoenfeld’s (1985a) book and subsequently using that information to reflect on their personal problem solving processes appeared have productive results as the semester progressed. As will be discussed in Chapters Six and Seven, it appeared that specific changes in Amy’s and Ben’s problem solving behaviors that emerged during this sequence of activities were subsequently sustained through the balance of the instructional sequence.

*Extended Analysis Tasks*

Dick Stanley of the Dana Center at the University of California, Berkeley introduced me to a set of tasks that had the potential to promote independent mathematical exploration in students. He called these tasks "Extended Analyses," and he developed some examples for use in professional development settings (see also Usiskin et al., 2003). In these tasks, participants are presented with typical textbook math problems from secondary mathematics curriculum and are guided through a process of
Suppose that on a round trip you go at 30 mph on the way out and at 60 mph on the way back. What is your average speed?

Figure 17: The Round Trip Problem.

The first step in the process is to find the numerical solution to the problem.

\[ D = RT \], so the rate is distance divided by time, \( R = \frac{D}{T} \). The time for the outbound plus the time for the return equals the total time using the average rate of speed, \( x \). The distance for the total trip is \( 2d \). Therefore,

\[
\frac{d}{30} + \frac{d}{60} = \frac{2d}{x} \Rightarrow \frac{2d}{60} + \frac{d}{60} = \frac{2d}{x} \Rightarrow \frac{3d}{60} = \frac{2d}{x} \Rightarrow 3x = 120 \Rightarrow x = 40
\]

To explore this problem more deeply, the next step would involve replacing specific constants with parameters. Let \( d \) = distance, \( r_1 \) = rate of the outbound trip, and \( r_2 \) = rate of the return trip. Again, the time outbound plus the time to return equals the total time using the average rate. Therefore,

\[
\frac{d}{r_1} + \frac{d}{r_2} = \frac{2d}{x} \Rightarrow \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{x} \Rightarrow r_2x + r_1x = x(r_1 + r_2) = 2r_1r_2 \Rightarrow x = \frac{2r_1r_2}{r_1 + r_2} \Rightarrow x = \frac{2}{\frac{1}{r_1} + \frac{1}{r_2}}
\]
\[ x = \frac{2}{\frac{1}{r_1} + \frac{1}{r_2}} \] is the harmonic mean of the speeds. From here, the problem can be further analyzed to investigate how each parameter impacts the problem context or situation. For example, one can fix one of the speeds and explore the resulting effect on the other speed: \( \lim_{r_2 \to \infty} x = \frac{2}{\frac{1}{r_1} + \frac{1}{r_2}} = 2r_1 \). So in the original problem, the average speed is limited to 60mph, regardless of how fast the return trip is. In this way, the learner would be guided to investigate the mathematical structure of the problem well beyond the numerical solution.

Previous iterations of the project revealed that PSMTs needed some initial instructor support if they were to be successful in their efforts to engage in these tasks (Bloom, 2007). As a result, I developed a protocol for introducing and supporting student work with Extended Analysis Tasks. For instance, when encountering the first Extended Analysis Task, students were walked through the steps of parameterizing and extending the problem. They were asked to search for specific results. The only step they were expected to complete independently was to create an isomorphic problem situation—that is, a new problem that has the same mathematical structure revealed in the analysis but is embedded in a different context. Throughout the semester, the scaffolding was gradually removed until students could be expected to take a problem and extend it on their own. Additionally, students made formal presentations of their explorations to their peers.

A total of five Extended Analysis Tasks were incorporated into the instructional design for the purpose of fostering mathematical curiosity and exploration.
Concluding Remarks

The purpose of this chapter was to provide details about the goals and motivation for the research design of the instructional sequence. It serves to further ground the study both in the literature as well as in my professional experience as a teacher, and it gives a context for understanding the data analysis that is put forth in the next two chapters.
CHAPTER SIX

THE CASE OF AMY

This chapter presents results from analyzing data that were collected from the case study of Amy. Results from analysis of Amy’s problem solving sessions are presented to describe shifts in Amy’s problem solving abilities as she progressed through 22 sessions of the instructional sequence. Data are presented to characterize Amy’s beliefs about mathematics, confidence in her mathematical abilities and persistence in working through complex problems. The Multidimensional Problem-Solving (MPS) Framework discussed in detail in Chapter Three was used to analyze this data and characterize these shifts. In particular, evidence of orienting, planning, executing, and checking are provided and discussed.

The first section characterizes Amy’s mathematical dispositions prior to the instructional sequence. Results from the content-focused pretest (Appendix A) provide information about her mathematical abilities with respect to high school content. Results from Amy’s completion of the Views About Mathematics Survey (VAMS) (Appendix B) are presented. Analysis of her pre-instruction interview is presented; this data provided information about Amy’s previous mathematical experiences and problem solving abilities.

The second section presents data from analyzing Amy’s written solutions to class assignments. It also includes analysis of Amy’s responses to in-class tasks that were completed in collaboration with her peers. Results from analyzing Amy’s responses to
the five Extended Analysis Tasks that were completed during the semester are also included.

The third section presents data from Amy’s performance on post-instruction measures. Shifts in Amy’s performance are reported by comparing pre and post instruction data.

The chapter concludes with a summary of insights gained from investigating Amy’s problem solving behavior.

*Prior to the Instructional Sequence*

Amy was in her junior year at the university when she enrolled in the experimental class. She anticipated student teaching and graduating within the current academic year. She had completed the three-course calculus sequence, Linear Algebra, Mathematical Structures and the Methods for Teaching Mathematics, and she was concurrently enrolled in courses required by the College of Education for a teaching certificate. During the experiment, she was also enrolled in Number Theory and College Geometry.

*Pretest Results*

Amy completed a pretest consisting of 30 items chosen from 2003 National Assessment of Educational Progress (NAEP) grade 12 items (see Appendix A for the pretest document). All items were multiple choice and all were categorized as *hard* according to NAEP documents. Amy answered 22 of the 30 questions correctly;
Table 4 shows the eight items she answered incorrectly. Her performance on this measure revealed that she was able to provide answers to standard high school mathematics problems. When compared to her classmates, Amy’s score ranked below the median (Refer to Appendix A for scores of other members of the class). Analysis of her incorrect responses revealed that she experienced difficulty with the items requiring knowledge of geometric structures (4, 6, 18, and 27) as well as the interpretation of rates-of-change (6 and 30).
Table 4

*Amy’s Incorrect Responses on the Pretest*

<table>
<thead>
<tr>
<th>Item number</th>
<th>Pre test item</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td><img src="image" alt="Image of triangles" /></td>
</tr>
<tr>
<td></td>
<td>If triangles ADE and ABC shown in the figure above are similar, what is the value of $x$?</td>
</tr>
<tr>
<td>6</td>
<td>A rectangular pool 24 feet long, 8 feet wide, and 4 feet deep is filled with water. Water is leaking from the pool at the rate of 0.4 cubic foot per minute. At this rate, how many hours will it take for the water level to drop 1 foot?</td>
</tr>
<tr>
<td>15</td>
<td>In a group of 1,200 adults, there are 300 vegetarians. What is the ratio of nonvegetarians to vegetarians in the group?</td>
</tr>
<tr>
<td>18</td>
<td><img src="image" alt="Images of cylinders" /></td>
</tr>
<tr>
<td></td>
<td>In the figures above, the radius and height of each right circular cylinder are given. If $w$, $x$, and $y$ represent the respective volumes of the cylinders, which of the following statements is true?</td>
</tr>
<tr>
<td>20</td>
<td>The least common multiple of 8, 12, and a third number is 120. Which of the following could be the third number?</td>
</tr>
<tr>
<td>23</td>
<td><img src="image" alt="Image of calculator display" /></td>
</tr>
<tr>
<td></td>
<td>The figure above shows the display on a scientific calculator. The value of the displayed number is between which of the following pairs of numbers?</td>
</tr>
<tr>
<td>Item number</td>
<td>Pre test item</td>
</tr>
<tr>
<td>-------------</td>
<td>---------------</td>
</tr>
<tr>
<td>27</td>
<td>[Diagram showing semicircles on an equilateral triangle] Semicircles are constructed on the sides of an equilateral triangle, as shown in the figure above. Of the following, which best approximates the sum of the lengths of the three darkened arcs?</td>
</tr>
<tr>
<td>30</td>
<td>A baseball card increases in value according to the function, $b(t) = \frac{5}{2}t + 100$, where $b$ gives the value of the card in dollars and $t$ is the time (in years) since the card was purchased. Which of the following describe what $\frac{5}{2}$ conveys about the situation?</td>
</tr>
<tr>
<td>I.</td>
<td>The card’s value increases by $$5$ every two years.</td>
</tr>
<tr>
<td>II.</td>
<td>Every year the card’s value is 2.5 times greater than the previous year.</td>
</tr>
<tr>
<td>III.</td>
<td>The card’s value increases by $\frac{5}{2}$ dollars every year.</td>
</tr>
</tbody>
</table>

$n=8$

Analysis of Amy’s performance in each of the five NAEP Content Areas\(^3\) (Table 5) revealed that she selected the correct response for only one of the four items in the content area of Measurement. She performed better on items assessing knowledge in the content areas of Algebra and Functions and Data Analysis, Statistics and Probability.

\(^3\) See Appendix A for information about the definitions of these content areas
Table 5

Amy’s Errors by Content Area

<table>
<thead>
<tr>
<th>Content area</th>
<th>Number of items in content area in pretest</th>
<th>Items missed by Amy in content area</th>
<th>Percentage of content area items missed in pretest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry and Spatial Sense</td>
<td>n=4</td>
<td>#4</td>
<td>25%</td>
</tr>
<tr>
<td>Number sense, properties, and operations</td>
<td>n=8</td>
<td>#15, #20 and #23</td>
<td>37.5%</td>
</tr>
<tr>
<td>Measurement</td>
<td>n=4</td>
<td>#6, #18 and #27</td>
<td>75%</td>
</tr>
<tr>
<td>Algebra and functions</td>
<td>n=10</td>
<td>#30</td>
<td>10%</td>
</tr>
<tr>
<td>Data analysis, statistics, and probability</td>
<td>n=4</td>
<td>all correct</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>30 total</td>
<td>8 incorrect responses</td>
<td></td>
</tr>
</tbody>
</table>

Table 6 contains Amy’s results clustered by NAEP Mathematical Ability\(^4\). She missed 43% of the questions in the area of Procedural Understanding, which NAEP defined as a student’s "ability to connect an algorithmic process with a given problem situation, to employ that algorithm correctly, and to communicate the results of the algorithm in the context of the problem setting" (National Center for Educational Statistics, 2001). This result suggested that in these problems, Amy was unable to access the mathematical resources required to solve the problem.

\(^4\) See Appendix A for information about the definitions of these ability types
Table 6

Amy's Errors by Mathematical Ability

<table>
<thead>
<tr>
<th>Mathematical Ability</th>
<th>Number of items in ability type in pretest</th>
<th>Items missed by Amy in ability type</th>
<th>Percentage of ability type items missed in pretest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem Solving</td>
<td>n=13</td>
<td>#6, #18 and #27</td>
<td>23%</td>
</tr>
<tr>
<td>Conceptual Understanding</td>
<td>n=10</td>
<td>#15 and #30</td>
<td>20%</td>
</tr>
<tr>
<td>Procedural Understanding</td>
<td>n=7</td>
<td>#4, #20 and #23</td>
<td>43%</td>
</tr>
<tr>
<td>30 total</td>
<td>8 incorrect responses</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In summary, analyses of Amy’s responses to the pretest items suggest that Amy’s mathematical content knowledge is not strong with respect to the abilities assessed by the NAEP items. Her responses in all content areas except Data Analysis, Statistics and Probability suggest that there may be deficiencies in her knowledge of those content areas, and her responses in the different ability types suggest that she failed to access appropriate mathematical knowledge for the problem setting.

Views About Mathematics Results

Amy completed the Views About Mathematics Survey (VAMS) at the beginning of the course. Amy’s responses were then categorized as expressing expert, mixed, or folk views in accordance with the classification scheme set out by Carlson, et al. (1998). Once the responses were classified, the number of responses in each category was counted to determine if the survey indicated someone espousing expert, upper transitional, lower transitional, or naive views about the practice, teaching, and learning of mathematics. Analysis of Amy’s survey revealed eight expert responses, seven mixed view responses,
and six folk view responses, placing her in the naïve classification. Table 7 contains the six questions revealing Amy’s naïve views about mathematics. Her response to #5 revealed her view that her job as a student is to replicate what her teacher does. A calculational orientation (A. Thompson, Phillip, & P. Thompson, 1994) was revealed by her responses to #10 and #19 in that these items suggest that she placed greater value on numbers and calculations than on ideas and relationships. Items #6 and #20 suggest that Amy did not have confidence in her mathematical abilities. Her response to item #24 suggests that she valued measurements and calculations over logical arguments. Carlson (1998) reports that "expert beliefs and high self-confidence appear to be desirable, but not easily acquired once “naïve views” are held" (p.25). Other studies about beliefs and mathematics (cf., Cooney et al., 1998; McLeod, 1992; A. Thompson, 1992) support the assertion that these views may be resistant to change.
### Table 7

**Items in Which Amy Demonstrated "Folk Views" About Mathematics**

<table>
<thead>
<tr>
<th>Item with responses by Amy categorized as &quot;Folk Views&quot;</th>
<th>Amy's response</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. My score on mathematics exams is a measure of how well:</td>
<td>6</td>
</tr>
<tr>
<td>(a) I understand the covered material.</td>
<td>{Expert View: options 1-4; Mixed View: option 5; Folk View: options 6-7}</td>
</tr>
<tr>
<td>(b) I can do things the way they are done by the teacher or in some course materials.</td>
<td></td>
</tr>
<tr>
<td>6. When I experience a difficulty while studying mathematics:</td>
<td>3</td>
</tr>
<tr>
<td>(a) I immediately seek help, or give up trying.</td>
<td>{Expert View: options 6-7; Mixed View: option 5; Folk View: options 1-4}</td>
</tr>
<tr>
<td>(b) I try hard to figure it out on my own.</td>
<td></td>
</tr>
<tr>
<td>10. Mathematical formulas:</td>
<td>5</td>
</tr>
<tr>
<td>(a) express meaningful relationships among variables.</td>
<td>{Expert View: options 1-3; Mixed View: option 4; Folk View: options 5-7}</td>
</tr>
<tr>
<td>(b) provide ways to get numerical answers to problems.</td>
<td>6</td>
</tr>
<tr>
<td>Item with responses by Amy categorized as</td>
<td>Amy's response</td>
</tr>
<tr>
<td>---------------------------------------------------------------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>&quot;Folk Views&quot;</td>
<td></td>
</tr>
<tr>
<td>{VAMS response ranges}</td>
<td></td>
</tr>
</tbody>
</table>

19. Collecting and graphing real world data is useful for:  
   - (a) determining patterns and making general predictions.  
   - (b) obtaining numerical answers to specific problems.  

   (Expert View: options 1-3;  
   Mixed View: options 4-5;  
   Folk View: options 6-7)

20. For me, making unsuccessful attempts when solving a mathematics problem is:  
   - (a) a natural part of my pursuit of a solution to the problem.  
   - (b) an indication of my incompetence in mathematics.  

   6  
   (Expert View: options 1-2;  
   Mixed View: option 3;  
   Folk View: options 4-7)

24. In order to prove a mathematical theorem one must:  
   - (a) produce evidence from the physical world.  
   - (b) provide a logically sound argument.  

   2  
   (Expert View: options 6-7;  
   Mixed View: options 4-5;  
   Folk View: options 1-3)
Overall, results from VAMS reveal that Amy held naive views about mathematics in that she valued numerical calculations over patterns, relationships and logical arguments, and was not confident in her abilities to solve problems.

_Pre-Instruction Interview_

The next pages report the results of an interview with Amy during the first week of the semester. The interview lasted approximately 65 minutes and was conducted by me (the researcher and instructor). Amy was asked a variety of questions about her previous mathematical experiences (refer to Figure 6) and was asked to solve three mathematics problems (refer to Figure 7) using a talk aloud protocol.

_Responses to the general questions_

During the interview, Amy was asked a series of questions about her previous experiences solving problems and studying mathematics. When asked about her interest in mathematics, Amy responded as follows:

I’m not an English person. I’m sort of a numerical [person]. I have, like I can make everything add up to 7 almost, but that’s something stupid, my phone number I can make to 7. I just do the mathematical operations.

In this passage, Amy revealed that she saw herself as a “numerical person.” This characterization together with her example suggests that for Amy, being good at mathematics is being good with numbers and numerical operations.

When asked, "What kind of math are you good at," Amy responded with the following:

[1] Amy: Honestly, I really prefer algebra over geometry but actually there is some geometry that I like and not the higher level stuff in
college. I don’t really care why, or, since it’s already been proven, I don’t want to prove it, I want to use it and help me solve stuff.


[3] Amy: I don’t know if that’s a bad attitude, but that’s what I, huh?


[5] Amy: I think calculus I would like if I had a good teacher. I’ve taken so far the three series and I had a good teacher at the very end which sort of helped to explain Calc I and Calc II in the 3rd version, but then I started to like it. Before that, I didn’t really pick up on it, so.

[6] Int: Because a lot of times algebra-kind-of-people really like calculus too.

[7] Amy: Once I had the Calc III part this teacher started explaining. Like, I didn’t understand why this was and he was starting to explain it.

In [1], Amy stated not only that she enjoyed algebra over other areas of mathematics, but her preference for algebra problems did not extend to the study of calculus [5-7]. Her statement, “I didn’t really pick up on it” suggests that she did not feel confident about her knowledge of calculus topics. These statements corroborate the results of Amy’s pretest. She performed much better on items classified in the content area of Algebra and Functions than she did on items in the areas of Geometry and Spatial Reasoning and Measurement. Pretest analyses also revealed that she experienced difficulty with problems involving rate-of-change. In statement [1], she said “I don’t really care why. . . I want to use it to solve stuff.” This suggests that she did not value
understanding structure or relationships in mathematics; she valued its utility for finding solutions. Her statement, “since its already been proven, I don’t want to prove it,” suggests that she did not value abstract representations such as theorems and did not value logical arguments as being important to her. It is interesting to note that these statements correspond to the naïve view of mathematics revealed in VAMS.

When asked what she thought of as problem solving, she said, “Pretty much there’s a situation we need to figure out, I don’t know.” When asked, she expressed confidence in her ability to solve problems.

I’m pretty logical, I mean, come to daily life problem solving, I’d be more of a logical sense person. Well, I came here because I have class after this, rather than come here earlier and have nothing to do. What else… I think through stuff.

She was asked about how confident she felt about her understanding of high school mathematics.

I feel pretty confident in it but right now I need to go back and review the stuff. I haven’t seen the stuff in forever. My brother is in high school and he’ll come up to me and I’m like, ‘Danny, honestly, I need to look at your book, I haven’t seen this forever, so I need to refresh my knowledge on some of this stuff like just, I mean like, there’s the way the book might teach you and there’s other short cuts that make sense that teachers would explain but I haven’t seen it forever so it has to come back to me with me working on it. I haven’t done high school math forever now.
In these two passages, Amy expressed confidence in her problem solving abilities and in her knowledge of high school mathematics, even though she stated “I need to refresh my knowledge on some of this stuff.” Her professed confidence is surprising because her pretest score was below the average in the class and revealed deficiencies in her content knowledge, and her responses to several VAMS items suggested that she did not possess confidence in her abilities. Earlier she suggested that she wasn’t confident in her knowledge of calculus. With these conflicting responses to questions about confidence, it is difficult to ascertain how confident Amy was in her abilities.

Amy recognized her shortcomings with respect to geometry and did not look forward to teaching that subject in high school.

Right now I don’t really want to teach geometry, but I think if I had more experience with it, I would. I took geometry in like a month in high school and that was it. And that’s pretty much the extent of my geometry knowledge. So I think if I had more, it would be better, but with geometry, it’s got the proofs and that’s the whole trying to explain “why” and so yeah, so mine would more be geometry. When I get a job, I’m going to tell them that I want to, I mean if they don’t give me the lower levels, I would want algebra, Algebra I-II, Algebra III-IV, that’s the area that I more enjoy.

This passage substantiates her claim that she is more of an “algebra person.” It might also explain her poor performance on items from the content areas of geometry and measurement on the pretest. In the next excerpt, Amy was asked about her experience in planning solutions.
Int: Ok. When you sit down to solve a problem, do you sort of, do you make a mental plan of what you’re going to do, or do you just start and go forward?

Amy: I try and relate it first, like if I have an example I’ve already done. And with past knowledge, otherwise I don’t just jump out there and go like try this way because if there’s nothing that leads me to that direction to think to go that way, then I won’t. If I am lost . . . I try to just fill in all the blanks.

The passage reveals that when planning a solution, Amy’s primary strategy is to relate it to a problem she has seen before. Her statement “I don’t just jump out there and go like try this way because if there’s nothing that leads me to that direction to think to go that way” suggests that if she had few tools available, leaving her to “try and fill in all the blanks”. This corresponds to her VAMS response of “My score on mathematics exams is a measure of how well I can do things the way they are done by the teacher or in some course materials.” In the next passage, Amy expressed her experience managing frustration.

Int: So what do you normally do when you’re feeling kind of stuck on a problem? What strategies do you have?

Amy: I would, I look at my brother’s high school book, figure out what the —

Int: So you check some resources.

Amy: For the formula.
[5] Int: How do you manage frustration? Are you more likely to put it aside and come back to it later or just bull through it until you figure it out?


[7] Int: Oh, just something like this.

[8] Amy: Something like this I think I could find if I, I don’t know if the formula would help me. I think that would give you the distance here.

In this passage, Amy revealed that when she was confronted with dead-ends while solving problems; her primary strategy was to look for formulas that might help her [4, 8]. This response supports earlier statements about the utilitarian value of mathematics.

Amy’s answers to the general questions revealed that the study of mathematics had value for its utility in finding numerical solutions to problems, and she did not “care why” something was true mathematically. Even though she professed confidence in her knowledge of high school mathematics, she offered contradictory statements that she did not feel confident in her understandings of calculus and geometry. She stated that she devised solution paths by trying to relate new problems to ones she had already seen, and if that strategy failed, she looked for formulas to help her. Here, responses to the general questions corroborated the results of her pre-instruction VAMS.

*The problem solving session*

During the problem solving portion of the pre-instruction interview, Amy was presented with three problems to solve. The first problem she attempted was the Square Paper Problem (Figure 18). As she worked on the problem, Amy spent a significant
amount of time trying to make sense of the problems although that effort did not appear to be useful for helping devise a viable solution.

A square piece of paper ABCD is white on the front side and black on the back side and has an area of 3 square inches. Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is 1/2 white and 1/2 black. How far is A' from the fold line?

**Figure 18:** The Square Paper Problem.

[1] Amy: Sure. First I draw the white side. Corner A is folded over, which lies on the diagonal BC. Ok. The way I am understanding it is that when you fold over the tip, it completely cuts it in half. [pause]

So, oh wait, no it doesn’t. It gives you half black, half white.

*She looks to interviewer for help*

**Figure 19:** Amy’s initial sketch of the problem.

[2] Int: So you’re going to fold the corner over

Again, she looks up questioningly; to make sure she has it right

[4] Int: Right. And then what do you see

[5] Amy: I don’t know how you get half black half white, because this diagonal they want it along. Because if this is A, this is C, they said they want the point to be on this diagonal, so then I mean, the only way I’m seeing this is, this is going to be half, even though these pieces aren’t the same, this area is ½, this is ½, is what I think they’re saying. How far is A’ from the folded line? So they want to know the distance from here. Alright, so back to this figure, we’re going to need to figure out the area of the square they said was 3 and then subtract it from the area of this, no. Figure out that new distance. [long pause as she looks at the paper]

In this passage, [1-5] Amy struggled to organize the information and make sense of the problem, actively seeking the interviewer's assistance. The following lines [6-9] highlight her difficulty devising a useable plan for solving the problem.
[6] Int: So what are you thinking?

[7] Amy: I’m trying to figure out how to get to that distance.

[8] Int: So what kind of options are you considering?

[9] Amy: First I was thinking of subtracting the triangle area but that gives us what this space is and that doesn’t say anything about the distance.


\[ \text{Figure 21: Amy's model for the folded paper.} \]

[11] Amy: This area is \( \frac{1}{2} \), this area is \( \frac{1}{2} \). This is a triangle, so \( \frac{1}{2} b \times h \), I don’t know if this helps me at all. This distance — I’m not sure the way to find the distance, like I think about going round about to fill in just holes. This we can say is a 90 degree angle, correct? And then since this is a square, this is a 45, 45, 90 — Is this square
root 2 and this 1, 1? Then special triangles, I don’t remember it.

30, 60, 90 is \( \sqrt{3}/2 \). Square has got an area of 3 which is 3 equals base times height. Since it’s a square, these two are the same, it could be like \( B^2 \). \( B = \sqrt{3} \) to give us the side of this.


[13] Amy: Because this is 1.5 something in length, 1 point something — less than 2. The one I’m stuck on basically, I remember this being 45, 45 90, this being radical 2 and this being 1 and 1. But why is that always 1? Because I could draw a triangle really big, it’s not just 1. That length wouldn’t be 1, same angles.

Amy spent over 13 minutes unsuccessfully trying to solve the Square Paper Problem, but she abandoned her solution when offered the opportunity. Her sketch in Figure 19 suggested that she was unable to make sense of the problem statement. Her strategy for solving the problem involved what appeared to be random attempts which did not seem to be based on a logical foundation. Her main strategy seemed to be "I think about going round about to fill in just holes" [11]. As mentioned earlier, she made a similar statement, “I try to just fill in all the blanks,” when the interviewer asked how she planned her solutions. Statements made during her attempt to solve the Square Paper Problem also suggested that she did not recognize the connection between the areas of the triangles and the distance she was trying to find [9]. She did not access knowledge of right triangle relationships to find the distance from the fold [11], nor did she appear to understand that the numbers associated with special triangles are ratios, not measurements [13]. This
revealed deficiencies in her conceptual understanding of the properties associated with special right triangles. There was also no evidence that she monitored the effectiveness of her thinking or strategies and gave little attention to the efficacy of her approach.

The next excerpt shows Amy’s work on the Least Common Multiples Problem (Figure 22).

List all the integers less than 100 that are divisible by 2, 3, 4, 5, and 6

Figure 22: The Least Common Multiples Problem.

Amy: So all the numbers that are divisible by these 5 numbers, correct?

Amy: To be divisible by 2, it’s got to be even. To be divisible by 3, the two numbers need to add up to 3, the sum is 3. To be divisible by 5, it needs to end in 5 or 0.

Amy: And I would go through and do trial and error. It’s not the quickest way, but. With products of 5, you basically have 5-100, 5, 10, 15, 20. All the ones in the left column are not even, so this makes it pretty easy. 50, 60… 80, so that takes care of the 5 factor. A sum of 3, you’ve got 30, 60, and 90. They’re all even and now I’ve just got to check if it’s divisible by 4 and 6.

Amy: 4 does not go into 30, 4 does go into 60, does 4 go into 90? So 90 over 4, no. And yes 6 goes into 60, so I guess there is only one integer less than 100 that is divisible by 2, 3, 4, 5, and 6, which is 60. Am I right?

Int: I’m not going to tell you.

Amy: I can’t know if I’m right?
Int: I can’t tell you if it’s right or not. Do you think it’s right?

Amy: The fact it’s got to be even, it has to end in 5 or 0 would be only those, I believe so.

Although it seemed that Amy made sense of the information in the problem statement, it appeared that she accessed only superficial knowledge about factors and divisibility to devise a solution. Amy used "guess and check" as her solution strategy. While she checked her calculations, she asked the interviewer if the answer was right, indicating that she did not trust her verification. As with the Square Paper Problem, it appeared that she did not reflect on the quality of her solution strategy, nor did she reflect on the reasonableness of her results until the interviewer failed to validate her solution.

Her work with the Mirror Numbers Problem (Figure 23) revealed a tendency to rely on calculations as a sense-making tool and “guess and check” as her primary solution strategy.

Two numbers are "mirrors" if one can be obtained by reversing the order of the digits (i.e., 123 & 321 are mirrors). Can you find:

(i) two mirrors whose product is 92565
(ii) two mirrors whose sum is 8768

*Figure 23: The Mirror Numbers Problem.*

Amy: Well, I’m going to do part (ii) first.

Int: Ok.

Amy: But it doesn’t look right. You need the sum to be 8,768 of, ah shoot.

It doesn’t say how big the numbers are. Ok… First, let me make sure I got
it right. If it was 123 and 321 it’s 123+321 is the sum, which is 444. Well, I thought it was two 4-digit numbers but that may not be true because … I was thinking that these two numbers [indicating the first and last digits] would be the same, which they are but these two [indicating the two digits in the middle] are not. So let’s see, I don’t get why these two won’t be the same as well in the middle.

[4] Amy: Maybe it’s a three digit number with a, no. Because if these two are still the same, I need to make the center one so it’d be different. So I’ve got my $ad$ and $da$ but how can I get a 6 on one side and a 7 on the other? I mean, I don’t know what $a$ and $d$ are yet but it could be 44, it could be 6, 2.

[5] Amy: I don’t think it really matters at this point to add up to 8 but to get a 7 and 6. 6 here with an extra 1, we need 16 but then that will add a 1 over here. That gives me 16, that gives me 17 but its going give me here… I mean I could have done 8 and 8 and still 16 and 17 but then how do I keep this one an 8 and that one an 8?

Figure 24: Amy's work on part (ii) of the Mirror Numbers Problem.
Int: So what are you thinking?

Amy: I’m thinking whatever numbers I pick for $a$ and $d$ in the 1’s digit, it
will be an 8. But in the 1,000’s it will be a 9 because if to satisfy the
middle numbers with a 6 and a 7.

Int: So you’re saying there’s no solution?

[Almost a full minute passes]

Amy: I'm trying to think if there’s another way to get a 6 and a 7 in here. I
don’t think so. Whatever two I pick here gives me an 8. Pick something
here to give me a 6 but to get a 7 here I’ll have to get a 1 here which will
give me a 1 here, which will yeah, make that 9. I think so. Yeah, I don’t
know actually. I thought that was the easier one.

Int: Ok, why don’t you try the first one then?

Amy: Okay.

In [3], it appeared that Amy engaged in sense-making behavior when she tested her
understanding of the term “mirror numbers.” Her initial plan appeared to be to determine
how many digits the solution would have [3, 4] and her statement, “I don’t get why these
two won’t be the same as well in the middle,” suggests that she was reflecting on the
reasonableness of her thoughts. She began to experiment with calculations on the
calculator, with little direction or purpose [4, 5]. Even when, in passage [8], the
interviewer offered the option of no solution, Amy did not appear to have the confidence
[9] to select that option or even offer an alternative solution. She quickly abandoned the
task after the prompt [11]. She went on to attempt part (i) of the same problem.
[12] Amy: Ok. So, see I thought this one was more difficult because I don’t know how big the number is.

[13] Amy: Well, I don’t know exactly what it is but I’m going to work on it like with that last problem, it ends in a 5.

[14] Amy: Umm, I guess that’s just a key factor, doesn’t mean anything. Not an even number, so the last number’s not 2 but, so we know 2 is not in the last digit.

[15] Amy: Ok. *(imputing numbers into a calculator)* I’m just throwing out a number and seeing what it gives me.

[16] Int: Ok. So what did you decide based on that number?

*Amy continued to try numbers with a calculator, noting if they were too big or too small*

[17] Amy: So I know it’s a 3-digit number, but. So then let’s see, I’ve got abc can you find — times bca equals this. Alright, with multiplication, order doesn’t matter. I don’t know if that [is important].

Figure 25: Amy's work on part (i) of Mirror Numbers Problem.

[18] Amy: Alright I finally got a 5-digit number.
[19] Int: Ok, what did you multiply together?

[20] Amy: 222, and with 333 it was a 6, so it’s in between those two. But see, I don’t know if this is bad; I know there’s other ways because I’m completely doing trial and error and I know there’s a formula. Oh, I wasn’t using any zeros. My whole logic could be off.

[21] Int: I didn’t catch that.

[22] Amy: Oh, I was never thinking about a zero, but no, yeah, I think because now is like, it would be a 3-digit number times a 2-digit number. Do you get what I’m saying?

[23] Int: Yeah.

Amy continued to do calculations, rejecting combinations as she went along.

[24] Amy: I divided the big number by 15 to see what it gave me, but that would have been wrong anyway, but — what am I thinking? It is a number, not multiples.

In [13] and [14], it appeared that Amy was organizing the information and accessing knowledge about divisibility, but her work shown in Figure 25 revealed that she had already made a critical error in her thinking by treating $a$, $b$ and $c$ as independent factors as opposed to digits in a three digit number. To solve the problem she began once again to randomly try out numbers [15]. She admitted that her strategy was trial and error [20], using previous trials (that were too big or too small) to inform the next, while stating “I know there’s a formula.” Her statement, “My whole logic could be off,” in [20] suggests that she was reflecting on her thinking. In [24] she realized the error in the way she was thinking about the problem by saying, “What am I thinking? It is a number, not
multiples,” indicating on-line monitoring. However, she was still unable to construct a solution, and she soon abandoned the task.

In both parts of the Mirror Numbers Problem, Amy appeared to orient herself to the problem, but her orienting behavior was superficial. Amy appeared to use numerical calculations to first make sense of the problem statement, and then later as she attempted to solve the problem through trial and error. As seen in Figure 24, she became confused by her notational choices. Even though it appeared that she was engaging in some monitoring behavior, there was no evidence that such monitoring was productive in terms of advancing her solution. She abandoned both parts of the Mirror Numbers Problem before she found a solution.

In summary, Amy’s problem solving session revealed that she was likely to begin calculations prior to fully orienting herself to the problem space, or devising a plan. She used "guess and check" as her primary heuristic. It appeared that Amy used numerical calculations to both help her make sense of the problem and as a solution strategy in the absence of some formula to use. This was especially evident in her work on the Least Common Multiples Problem and the Mirror Numbers Problem. Analyses of the problem solving session revealed little evidence of monitoring.

The quality of Amy’s conceptual knowledge of mathematics appeared to be a source of difficulty for her in her problem solving attempts. The problem solving session also revealed that Amy was easily frustrated by failed solution attempts and readily abandoned tasks. She was able to find a solution to one of the three problems presented. These observations are interesting in light of her earlier statements that she felt confident
in her problem solving abilities and saw herself as someone who could "think things through."

This data together with her pretest and VAMS scores suggested that Amy experienced difficulty accessing appropriate mathematical knowledge when confronting novel problems. She did not appear to possess the conceptual or procedural knowledge needed to solve these problems. She held a naïve view of the methods of doing mathematics, and she was easily frustrated, often leading her to abandon her efforts. Her performance on all three of the measures (pretest, VAMS, interview) indicated that she is one of the weakest students in the class.

**Following Amy Through the Semester**

This section provides a description of Amy’s progress and participation over the semester, including her individual work as well as explanations and interactions with her group members, other classmates, and instructor. The research data has been organized into three sections, the first 5 sessions, the middle 10 sessions, and the final 8 sessions, to allow for investigation over shorter periods of time as well as across the entire semester.

*Early in the Semester: Sessions 1-5*

During this first phase of the semester, Amy’s mathematical behavior was similar to that revealed in pre-instruction measures.

*Problem sets*

Amy’s work on the problem sets showed many of the same difficulties revealed in the pre-instruction problem solving session. She struggled to identify key concepts used to solve the problem. Many of the problems she submitted during this segment of the
instructional sequence were incomplete, suggesting that she was not persistent in her efforts to find solutions.

Analysis of her homework in this early phase suggests that Amy's conceptual knowledge hindered her efforts to construct viable solutions. For example, when presented with the Square Peg Problem (Figure 26) Amy wrote that her strategy was to “find the difference in space left between” (Figure 27). Yet, errors in the mathematical statements she had constructed made such a comparison impossible for her.

5. Which leaves more open space – a round peg inside a square hole, or a square peg inside a round hole?

Figure 26: The Square Peg Problem from Problem Set 2, #5

Figure 27: Amy’s work on the Square Peg Problem
When Amy solved the Rug Problem (Figure 28), her solution used the formula for the area of an ellipse. Although the problem also required a solution using calculus, she wrote, "I couldn't think of a way to find the red border other than using the area formula."

1.) I have an elliptical shaped braided rug. The rug is 12 feet long and 6 feet wide. The rug is all brown except it has a red border that is 1 foot wide. What is the area of the red border? Show that you can find the solution using calculus.

*Figure 28: The Rug Problem from Problem Set 5, #1.*

In her solution of the Employee Rate Problem (Figure 29), Amy correctly calculated $N(4)$, but in part b, Amy calculated an average rate of change over the interval from $N(0)$ to $N(4)$, when the instantaneous rate of change was indicated in the question.

2.) A company estimates that in $t$ years the number of its employees will be $N(t)$, where $N(t) = 1000(0.8)^t$.

a.) How many employees does the company expect to have in 4 years?

b.) At what rate is the number of employees expected to be changing at 4 years?

*Figure 29: The Employee Rate Problem from Problem Set 2, #2.*

The episode below illustrates her reliance on memorized procedures. Amy made an unfounded assumption [5] about the shape of a rectangle in the Highway Fence Problem (Figure 30).

2.) A certain farm allocs $1000 for fencing a rectangular area that is to abut a highway. Because the fencing on the highway side must be attractive, it costs $4 per lineal foot. The other three sides of the area are fenced at $2 per foot. What are the dimensions of the rectangle that maximizes its area?

*Figure 30: The Highway Fence Problem, Problem Set 4, #2.*
Amy: I don’t see why I got the wrong answer. I wrote two different equations. You have a thousand dollars equals 4 times $y$ plus 2 times $s$, plus 2 times $s$ because those are all the sides.

$1000=4y+2s+2s+2s$

Inst: Uh huh.

Amy: And then I had a thousand equals 6$y$ plus 4$x$.

$1000=6y+4x$

Inst: Uh huh.

Amy: And then…what did I do then? [pause] Oh, since I decided it would be a square that would maximize the —

Inst: That was the assumption that you can’t make.

Amy: But that’s true.

Inst: That a square can maximize the area, but we have a constraint here based on the costs of the different sides. So we want to know how we can maximize the area for a thousand dollars. So that changes the problem. So you can’t just make that jump that it’s going to be a square.

It appeared that based on her prior experience, Amy assumed that a square would always produce a rectangle of greatest area [5]. She failed to take into consideration the other constraints in the problem [8]. In her orientation to the problem, Amy was quick to categorize it as a particular type of problem, but she failed to attend to all the information provided. Amy’s work on the Rug Problem, the Employee Rate Problem and the
Highway Fence Problem suggests that she experienced difficulty making use of the concepts learned in calculus, even when explicitly directed to.

In summary, the written work submitted during this instructional period suggests that Amy’s mathematical conceptions impeded her efforts to construct viable solutions to the problems presented. She especially seemed to struggle in instances where the concepts of calculus were indicated. Although she drew some diagrams, her primary heuristic (when she did not have a formula to use) was guess and check. In the five problem sets Amy submitted during this time, 10 problems were turned in unfinished or not attempted. This suggests that she was easily frustrated when she hit "dead ends" in her solution paths and lacked the tools (resources and heuristics) to find another way to solve the problems.

Class work

During the first class session, Amy appeared to be an interested student, asking many questions about the format of the class and the teacher’s expectations for written work. The following episode is from her presentation of her group’s solution to the Magic Square Problem (Figure 31).

| In your group, create a 3x3 “Magic Square” that uses the numbers 1, 2, 3, 4, 5, 6, 7, 8, & 9. |

Figure 31: The Magic Square Problem.

[1] Inst: So someone tell me how their group decided on what number each column and row had to add up to. So this group, how did you decide?
Amy: Well I personally figured out that the number had to be between 12 and 18 by adding up the two smallest numbers of the biggest and then the two highest numbers of the smallest. And 15 was in the middle and that’s how I figured out the number. But that’s pretty much how you guys figured out the other way, it was, they all add up to 45 and there is [sic] three columns so, they add up to 15.

In this passage, Amy explained how she used numerical calculations to find an interval (between 12 and 18) for the sum of each column. This behavior is similar to that revealed in the pre-instruction problem solving session, when Amy used numerical calculations to both make sense of the problem and guide her to a solution using guess-and-check methods.

The following excerpt occurred when Amy presented her group's construction of the Inscribed Circle Problem (Figure 32).

You are given two intersecting straight lines and a point $P$ marked on one of them as shown below. Show how to construct, using a straight edge and compass, a circle that is tangent through both lines and has the point $P$ as one of the points of tangency to one of the lines.

Figure 32: The Inscribed Circle Problem.
Amy: OK. What we did first, here’s point $P$ and we started at the vertex to find the other point of tangency down here. So we drew a line right here to that point $Q$. So then we found . . .

*Amy demonstrated by placing the point of her compass on the vertex of the angle and drawing an arc that intersected the ray at point $Q$*

Inst: Wait a second. Now how did you know to do that? I’m just curious. How did you know they were going to be the same length?

Amy: What do you mean?

Inst: Well how did you know that one point was going to be the same distance off the vertex of the angle as the other point?

Ben: When we put that circle in there?

Inst: It’s the way to fit that circle in there…

Amy: Because it’s got to fit a tangent point here and a tangent point here.

Inst: We’ll come back to that question. Go ahead.

Amy: OK. So then starting up here with $P$ we’re going to find the perpendicular line that runs through point $P$ and, umm, do you want to know how? I’ll show it over here. So we found the two perpendicular things and drew the line. And then we did the same, there’s other ways to do it, but we found the perpendicular line through $Q$ as well. And then that point of intersection gave us the
center of the circle, plus two perpendicular lines, bisectors will create that center point.

[10] Inst: Well if they’re both the radius then they will meet in the center of the circle, right?


[12] Inst: That’s what you meant, right?


[14] Inst: Exactly. So did everything that they — was everything they did justified?

Amy presented her solution by reviewing her actions [1, 9]. When the instructor asked her to justify her steps [2, 4, 10], Amy appeared unsure of how to respond [3], and even with the help of the instructor, she couldn't justify the first step of her construction [8, 9]. This episode revealed that Amy experienced difficulty when asked to mathematically justify procedures and corroborates the data from her pre-instruction interview when she stated that she was not interested in “the why” of mathematics.

In summary, Amy’s contributions during class revealed that she was comfortable using numerical calculations to help her make sense of the problem situation and to advance her solution attempts using guess and check strategies. Amy also struggled to justify her actions using logical mathematical arguments.

Extended Analysis Tasks

The first Extended Analysis Task assigned to the class involved the Catching up Problem (Figure 33).
Person A sets out in a car going at 50 mph. Starting 3 hours later, person B tries to catch up. If person B goes at 75 mph, how long does it take to catch up?

*Figure 33: The Catching up Problem.*

After finding a numerical solution for the problem, students were expected to parameterize the problem, explore the structure, and create an isomorphic problem – a problem that had the same mathematical structure as the Catching up Problem. The numerical solution for the problem, as well as some of the initial analysis, was conducted in class with the support of the instructor.

The take-home portion of the task asked the students to find the catching up time \( t \) as a function of the difference in the velocities. Rather than construct a function to model the situation, Amy first created a table, again relying on numerical calculations to make sense of the problem (Figure 34). She also used the table to draw conclusions about the behavior of the function.

<table>
<thead>
<tr>
<th>Car A (mph)</th>
<th>Car B (mph)</th>
<th>B - A</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>75</td>
<td>-125</td>
</tr>
<tr>
<td>100</td>
<td>150</td>
<td>-100</td>
</tr>
<tr>
<td>150</td>
<td>225</td>
<td>-75</td>
</tr>
<tr>
<td>200</td>
<td>300</td>
<td>-50</td>
</tr>
<tr>
<td>250</td>
<td>375</td>
<td>-25</td>
</tr>
<tr>
<td>300</td>
<td>450</td>
<td>0</td>
</tr>
</tbody>
</table>

*Figure 34: Amy's solution to the Catching up Problem.*
Once she made sense of the problem quantitatively, she then found $t$ as a function of the difference in the distance between Car A and Car B (Figure 35) but failed to interpret this result.

![Equations](image)

*Figure 35: Amy's attempted extension of the Catching up Problem.*

The isomorphic problem Amy submitted did not share the same mathematical structure as the Catching up Problem. She wrote:

> The second hand of a clock is 10.2 cm long. Find the linear speed of the tip of this second hand.

This simple linear function did not have the same mathematical structure as the original problem, and she made no attempt to prove her claim that it did have the same structure.

Her work on the extension assignment suggested she was obstructed in her efforts to complete the assignment by her lack of conceptual understanding of the functions involved. Her submitted work revealed that she needed to create a table of numerical calculations before she was able to construct the new function $t(w)$, suggesting she had difficulty making sense of the function and its relationship to the original problem. Her
submitted isomorphic problem suggested that she lacked sufficient understanding of the meaning of that function to create a similar problem.

**Summary of sessions 1-5**

During this early phase of the course, Amy’s problem solving behavior was observed to be similar to that revealed in the pre-instruction data. She experienced difficulty accessing appropriate mathematical knowledge when attempting to solve problems. Her impoverished knowledge base impeded her ability to make sense of problems, construct viable solution paths, and overcome failed efforts. She used few heuristics and often resorted to numerical calculations to both make sense of and find solutions for problems. These deficiencies led to her being easily frustrated and giving up on problems. In addition, there was no evidence that she monitored her work or her thinking during this segment of the instructional sequence.

**Transitions: Sessions 6-15**

In class sessions 6 through 15, changes in Amy’s mathematical behaviors became evident, particularly in the areas of planning and monitoring. Changes in her work with the assigned problems sets, her interactions during class, and her work on Extended Analysis Tasks are discussed in the following sections.

**Problem sets**

Although Amy's homework scores steadily improved, her conceptual and procedural understanding continued to impede her efforts when completing assignments during this segment of the instructional sequence. As an example, her work on the Quadratic Family Problem (Figure 36) again exhibited her reliance on numerical calculations as a problem solving tool.
Consider the family of quadratic functions given by
\[ f(x) = (m^2 - m + 1)x^2 - (2m)x + 1 \] where \( m \) is any real number. For what values of \( m \) will \( f(x) \) always be positive?

*Figure 36:* The Quadratic Family Problem from Problem Set 6, #2.

Rather than use analytical methods to find the interval of values for \( m \) that would make the function positive, Amy substituted 1, 0 and -1 for \( m \). Her calculations led her to an incorrect solution. Her work suggested that when she oriented herself to the problem, she did not attend to the meaning of “\( f(x) \) always positive”.

When solving the Remainder Problem (Figure 37) she accessed her knowledge of number theory and chose to use modular arithmetic to solve the problem (Figure 38). However, she did not understand modular arithmetic well enough to correctly set up and solve the problem.

4. When an integer is divided by 15, the remainder is 7. Find the sum of the remainders when the same integer is divided by 3 and by 5,

*Figure 37:* Problem Set 8, #4.

It appeared that, when Amy oriented herself to the problem, she focused on the word “remainder” rather than making sense of the question, and then she attempted to employ a memorized procedure.

*Figure 38:* Amy's solution for Problem Set 8, #4.
In her solution to the Radioactive Decay Problem (Figure 39) Amy interpreted the statement as representing a linear rather than exponential relationship.

1.) The rate of decay of a radioactive substance is proportional to the amount of substance present at any time \( t \). In 1840 there were 50 grams of the substance and in 1910 there were 35 grams. To the nearest gram, how many grams of the substance remain in 1990?

*Figure 39:* The Radioactive Decay Problem from Problem Set 9, #1.

There were instances when Amy demonstrated that she could justify her reasoning. When she presented her solution to the Double Triangle Problem (Figure 40) she justified the reasoning she used to try to find the length of \( AB \):

5. Points \( A, B, C, \) and \( D \) lie on a line, in that order, with \( AB = CD \) and \( BC = 12 \) \( BE = CE = 10 \). The perimeter of \( \triangle AED \) is twice the perimeter of \( \triangle BEC \). Find \( AB \)

*Figure 40:* The Double Triangle Problem from Problem Set 5, #5.

[1] Amy: With the given information I was able to find the perimeter of the outside of the triangle by doubling the smaller one. And then I set up an equation so that, on the bottom, the two sides next to the 12, I had \( 2x \) because they’re congruent plus 12 and then I came to a conclusion that the sides on the outside from \( A \) to \( E \) and \( D \) to \( E \) were equal because they’re both going to the point \( E \) and on the bottom from \( A \) to \( B \) and \( C \) to \( D \) it’s the same distance so I think they’re equal so I did \( 2x \) plus \( 2y \) plus 12 equals 64. And then I
simplified $x$ plus $y$ equals 26 and I know that $y$ is the hypotenuse so it has to be bigger than 10 or it has to be the longest side so it’s somewhere between 14 and 25 but I don’t know how to solve for it.

[2] Inst.: OK. So we’ve brought up some interesting issues. Do $AE$ and $AD$ have to be equal? In other words does triangle $AED$ have to be isosceles — the way the problem was written? Class agrees

In this passage, Amy was able to justify her assertion that $\Delta ADE$ was an isosceles triangle. Unable to exploit that fact, she performed calculations in order to find an interval of possible solutions (between 14 and 25). This strategy was not helpful because the solution required the use of the heuristic, "add an auxiliary element." That is, the altitude for $\Delta ADE$ needed to be constructed first, in order to make use of isosceles triangle properties. Amy did not use this heuristic.

4.) Graph the following equation and compute the area it encloses.

$$|2x - 10| + |5y - 10| = 20$$

*Figure 41:* The Absolute Value Problem from Problem Set 6, #4.

Despite these difficulties, there was evidence that Amy’s problem solving abilities were improving. For instance, when solving the Absolute Value Problem (Figure 41), Amy accessed knowledge of absolute value equations and correctly considered the four different cases, then graphed the correct solution.
A certain water lily grew extremely quickly and it doubled its surface area at the end of each day. At the end of the 30th day, it had entirely covered the pond in which it lived. If a second lily, identical to the first, has been in the pond, how long would the two lilies have taken to cover the entire pond?

*Figure 42:* The Water Lily Problem from Problem Set 10, #4

Amy’s solution to the Water Lily Problem (Figure 42) demonstrated that she was able to reason through the problem using logical arguments as opposed to numerical calculations. It appeared that her diagram (Figure 43) helped her make sense of the exponential growth described in the problem.

*Figure 43:* Amy’s work on the Water Lily Problem

By the end of class session 15 (the end of this instructional segment), Amy turned in a problem set (Problem Set 10) for which all problems were correctly solved with valid supporting mathematical arguments.

During this transition segment of the semester, Amy’s submitted assignments revealed some improvement in her problem solving behaviors. Although her content knowledge continued to hinder her efforts to construct solutions, Amy demonstrated that she could devise valid solution paths and support them with mathematical arguments. Amy continued to use guess-and-check as a heuristic; however she used other heuristics
as well, including that of examining cases in the Absolute Value problem and making
diagrams in the Water Lily problem. Amy submitted fewer unfinished problems during
the first five class sessions, suggesting an improvement in her ability to persevere in her
efforts to solve the assigned problems.

*Class work*

Amy's behaviors during group problem solving sessions revealed yet other
important aspects of her problem solving behavior. The class was presented with the
Triangle Problem (Figure 44) during the sixth class session. This difficult problem was
the focus of small group work for the next four class sessions.

You are given a fixed triangle $T$ with base $B$ as below. Show that it is
possible to construct, with a straightedge and compass, a straight line that is
parallel to $B$ and that divides $T$ into two parts of equal area.

![Figure 44: The Triangle Problem.](image)

Amy contributed little at first, listening instead as members of the group claimed
that to solve the problem, they needed to construct one of the centers of the triangle.

Soon, Amy began to question her peers about their arguments. For instance, she asked
Shania, “What are you doing with perpendicular bisectors?” Later that same session, she
challenged Shania's claim that all the triangles formed by the construction of the
orthocenter (Figure 45) were congruent.
[1] Amy: When you say they are congruent, do you mean, like, the same areas?

[2] Shania: No, they are the same. Same everything.


Figure 45: Amy's sketch of the orthocenter of a triangle.

Throughout the next few class sessions, Amy insisted that her group members explain their thinking to her. This behavior suggested that Amy was monitoring others' processes and products, exhibiting some mathematical integrity with respect to the claims made in the group. This behavior also revealed her effort to follow and make sense of others’ reasoning.

It is interesting to note that Amy's habit of using calculations to make sense of mathematical situations helped her convince the group that looking for the centers of the triangle was an unproductive solution path. In class session seven, Amy drew a "3-4-5" triangle [1] and constructed a line parallel to the base that passed through the orthocenter in order calculate the areas of the two resulting shapes.

[1] Amy: I have a 3-4-5 triangle

[3] Amy: Yeah, but it should still work the same, right? But it'll have nice numbers.

[4] Callie: To try and see —

[5] Amy: To figure out what we are saying. We're testing the vertex?


[7] Amy: So I have to draw that line.

*The instructor comes to the table*

[8] Callie: (To instructor) We're testing it with numbers.

In this passage, Amy made use of the heuristic "solve a simpler problem" to test the group's conjectures [5], indicating that if their plan worked, it would work for this particular triangle [3]. When she subsequently calculated the two areas, they were not equal. Convinced, the group was able to abandon their original solution strategy and consider alternate solution paths.

Throughout work on the Triangle Problem, Amy consistently made efforts to understand how others in the group were thinking. Even when they were finishing the construction of the line, Amy was heard asking Callie, "Okay, but why did you want to put the perpendicular there?" It appeared that her curiosity motivated her to want to understand the actions that others took.

When asked to reflect on her experience with the Triangle Problem, Amy showed insight into her problem solving behaviors when she wrote the following:

As I work on problems in the future, I will try and think about the knowledge I already have and how I can use that to solve the problem. . . .
So I guess you have to dissect what you know and then see what you need to solve and how you can do that with the pieces you know.

In this statement, Amy acknowledged her need to expend more effort in the orienting and planning phases of problem solving. These statements are particularly encouraging when compared to statements made during her pre-instruction interview in which she admitted she rarely thought a problem through, but rather she thought "about going round about to fill in just holes".

In rhombus EFGH the coordinates of E and G are \((-6,-3)\) and \((2,5)\) respectively. Find the area of the rhombus if the slope of segment EF is 2

\[\text{Figure 46: The Rhombus Problem}\]

Amy expressed more confidence in her knowledge of geometric structures as well. While solving the Rhombus Problem (Figure 46), when Ben appeared to be unsure of the properties of parallelograms and rhombi, Amy supplied them for him. She continued to press others for explanations. In the following passage, she asked Rolf to justify his conjecture about the slope of a line.

[1] Amy: How do you know that’s 45°?

[2] Rolf: Because that’s slope is 1 and that’s slope is 2, so —

[3] Amy: But how do you know that?

[4] Rolf: (pause) Because this is 2, and this is…wait. Oh, it’s not 45.

Because Amy pressed him to explain his reasoning \([1, 3]\), Rolf became aware that his thinking was flawed \([4]\). In this episode, it appeared that Amy’s sense making questions had the effect of motivating Rolf to reflect on the correctness of his mathematical constructions.
Amy’s engagement during group problem solving sessions revealed changes in her problem solving behavior. Her apparent desire to understand the thinking and constructions of others prompted her to ask sense making questions such as “How do you know that?” from the passage above. Written reflections revealed that Amy became aware that she needed to focus more on the orienting and planning phases of the problem solving cycle.

*Extended Analysis Tasks*

During class sessions 6 through 15, students completed and presented two Extended Analysis Tasks. They first extended the Evaporation Problem (Figure 47). This classic problem yields the surprising answer that 50.05% of the water has evaporated. Amy experienced difficulty assessing the relationships indicated by the problem statement.

A substance is 99% water. Some water evaporates, leaving a substance that is 98% water. How much of the water has evaporated?

*Figure 47: The Evaporation Problem.*


[4] Callie: No, it’s not that easy, is it?

[5] Amy: Why not? Because, lookit [sic]. If this 99% is water — and so I drew that picture. Then it evaporates and 98% is water, and the 2% here is what evaporated from the 99. So, .99 times 2….That’s interesting.
Amy determined that the solution was simply 98% of 99 [1-3], but when she explained her reasoning to the group, they were unconvinced [4]. In [5], Amy checked her thinking on a calculator and then appeared to abandon her argument. Even though her conceptual understandings led her to oversimplify the problem, it appeared that reflection on her explanation combined with some mathematical verification convinced her that her thinking was wrong. As the group continued to work on the problem, it appeared that Amy was unable to make sense of the proportional relationships without the support of Ben (and to a lesser degree, Callie). However, as with the Triangle Problem, Amy’s sense making questions and requests for justification had the effect of prompting others in the group to reflect on the progress and products of their problem solving.

In Amy’s extension of the Evaporation Problem, she correctly parameterized the problem, but her subsequent extension was limited to exploring the most extreme cases: when *none* of the water evaporated, and when *all* of the water evaporated. Since these two cases are trivial, she did not appear to gain any insight as to why the problem yielded such unexpected results. She did, however, take the opportunity to use the construction of an isomorphic problem to make sense of a previously puzzling problem. She wrote the following about her isomorphic problem:

> When I first read the problem, it made me think of a previous homework problem when they asked, do women make 25% or 33% less than men. Percentages can be quite confusing because you have to conceptualize different units. An isomorphic problem that came to mind when dealing with percentages is that I never understood how carpool lanes on the highway can cut traffic by 50%.
Amy was able to construct an appropriate isomorphic problem, and she demonstrated that it had the same mathematical structure as the original problem.

The second analysis assigned during this period was the Completing the Square Problem (Figure 48).

For the function \( f(x) = 6x^2 + 5x - 7 \), complete the square to name the vertex of the function as well as the \( x \) intercepts.

**Figure 48:** The Completing the Square Problem.

By parametrizing the problem, Amy correctly derived the quadratic formula and the general formula for the vertex of a parabola. During her class presentation, she appeared to be excited by her discovery. For her extension, Amy wrote, "I decided to look at the cases when \( a, b \ & c = 0 \) and see what the value of \( x \) will be." This appeared to be a genuine exploration that yielded (for her) unexpected results. In her work below (Figure 49), she created a table that summarized her results from substituting zero for one of the constants in the parameterization. The center column contained her results when substituting zero for \( a, b, \) or \( c \) directly into the quadratic formula, \( x = \frac{-b}{2a} \pm \frac{\sqrt{-4ac + b^2}}{2a} \), and then simplifying the result. The right column displayed the results of substituting zero for \( a, b, \) or \( c, \) into the equation, \( ax^2 + bx + c = 0 \), and simplifying that result first, and then finding the solution using the quadratic formula.
Figure 49: Amy's extension of the Completing the Square Problem.

Even though Amy’s problem solving behaviors were improving; this example suggests that she experienced difficulty relating the different representations of the quadratic she created. For instance, she did recognize the equivalence of the two expressions resulting from substituting \( a = 0 \). That is, she did not see that when \( a = 0 \), the function is no longer quadratic making the quadratic formula inappropriate.

Figure 50: Amy's reflection on her presentation.

She also did not see that the second result was simply the \( x \) intercept of the linear function resulting from the leading coefficient being zero. It apparently did not occur to her to investigate whether the solutions she found when \( b = 0 \) were equivalent (which
they are). Her presentation to the class generated a discussion about the meaning of the different representations involved. After her presentation, she wrote her understanding of what was discussed (Error! Reference source not found.). Her statement in this reflection about the result of substituting zero for the constant \( a \) suggests that there continued to be deficiencies in her understanding of quadratic function and its various representations.

Amy’s engagement with these two Extended Analysis Tasks revealed changes in her mathematical behaviors. Amy continued to exhibit the sense making and questioning behavior first observed during the solution of the Triangle Problem. Her extension of the Evaporation Problem revealed that she was able to construct an isomorphic problem and provide justification that it shares the same mathematical structure as the original problem. As mentioned earlier, in the Catching up Extended Analysis Task, Amy was unsuccessful in her attempts to create a problem that was isomorphic to the Catching up problem. In her extension of the Completing the Square Problem, it appeared that Amy made discoveries about quadratic functions that she found exciting.

It should be noted, however, that Amy’s mathematical conceptions interfered with her work on both of these problems. In the Evaporation extension, it appeared that her understanding of proportional relationships did not support her attempt to construct a solution for the original problem. In addition her content knowledge appeared to limit her avenues of exploration as the path she chose essentially trivialized the problem. In the Completing the Square extension, deficiencies in her understanding of quadratic functions were revealed by her difficulties interpreting different representations of the quadratic formula.
Summary of sessions 6-15

During this segment of the semester, some changes in Amy’s problem solving behavior became evident. Her assignments revealed improvement in her ability to construct productive solution paths and valid mathematical arguments. Amy was observed using heuristics other than guess-and-check, including examining cases and creating sense-making diagrams. It also appeared that Amy was more persistent, as evidenced by fewer unfinished homework problems. In her group interactions, she was observed working hard to both follow and reflect on the work of her peers. Through her work on the Triangle Problem, she expressed the realization that she needed to expend more effort in the orienting and planning phases of problem solving. Amy was able to construct isomorphic problems through her work on the Extended Analysis Tasks, and showed excitement and curiosity in her efforts to extend the problems.

Amy’s conceptual understanding of the mathematics at hand continued to be her greatest obstacle though it appeared she was motivated to enrich her mathematical conceptions.

Final Phase: Sessions 16-22

During class sessions 16 through 22, Amy’s problem solving behavior continued to evolve. Changes in her work with the assigned problem sets, her interactions in the class setting, and her extended analyses are discussed in the following sections. In particular, Amy’s use of a broader array of heuristics as well greater attention to the orienting phase of problem solving appeared to support her efforts to construct viable solution paths. Her final project and her final written reflections are also discussed.
Problem sets

Amy’s scores on problem sets continued to improve as most of the problem sets she submitted contained both correct solutions and appropriate mathematical justifications. She submitted only three unfinished problems during this time, demonstrating that she was more able to persist in her problem solving efforts. Amy also exhibited the ability to solve problems requiring geometric and analytical reasoning, two areas which presented difficulties for her in the past. For example, she constructed the following proof (Figure 52) to the Divisible by Eight Problem (Figure 51).

4. Select any odd number. Square it and subtract 1. Prove that the result will always be divisible by 8:

Figure 51: The Divisibility by Eight Problem from Problem Set 16, #4.

Figure 52: Amy's proof the Divisibility by Eight Problem

Although Amy’s notational choices were unsophisticated, she effectively employed the heuristic of considering different cases to construct a sound argument.
Amy’s submitted problems sets provided evidence that she employed other heuristics as well. In the logarithmic equation shown in Figure 53 she graphed both functions to determine the intersection.

\[ \log x^2 = \log(x^2). \]

*Figure 53: Problem Set 13, #2.*

Her graph (Figure 54) was used both as a sense making tool and led her to a correct solution. Later she was heard telling Callie, “You’ve totally taught me to graph. I never used to graph things.” It is worth noting that many of those PSMTs who used analytical methods to solve this problem failed to arrive at a correct solution.

*Figure 54: Amy’s solution of Problem Set 13, #2.*

When solving the Summation Problem (Figure 55) Amy noticed that pattern formed (Figure 56) when the subtraction was taken term by term.

\[ \sum_{n=1}^{80} (2n - 1) - \sum_{n=1}^{80} 2n \]

*Figure 55: The Summation Problem from Problem Set 14, #4.*

In addition to her use of heuristics other than guess-and-check, it appeared that Amy was more effective in her efforts to orient herself to the problems. Her work on all three of these problems suggest that she was taking the time to organize and make sense of the information provided, and using that to select a solution path.
During this period, Amy’s questions about homework problems extended from concern only about finding a solution to curiosity about other (possibly better), solutions. In class session 17, Amy presented a correct solution of the problem shown in Figure 57, where she graphed the left and right sides of the inequality as functions to locate the correct intervals.

\[
\begin{align*}
2. \text{ Solve } & \frac{x - 2}{x + 3} < 4 \\
\end{align*}
\]

Figure 57: Problem Set 12, #2.

After she presented her solution, she asked the instructor, “So then algebraically, how can you do that? Because you can’t multiply by \(x + 3\) because you don’t know if you need to flip the inequality or not.” Her previously mentioned comment to Callie, “You’ve totally taught me to graph. I never used to graph things,” also suggested a growing interest in alternative solution approaches.

Despite these gains in her problem solving behaviors, Amy’s mathematical understandings still obstructed her efforts to construct viable solutions. As an example, consider her solution to the problem in Figure 58.

6.) Without using a calculator, rank the following from greatest to least: \(e^e, \pi^\pi, e^\pi, \pi^e\) and explain your reasoning.

Figure 58: Problem Set 11, #6.
It appeared that Amy’s understanding of exponents and the numbers represented in the problem led her to oversimplify the problem. In her solution (Figure 59) she rounded $e$ to 2 and $\pi$ to 3, then claimed that it was reasonable to use $2^3$ as an estimate for $e^\pi$ and $3^3$ as an estimate for $\pi^e$. By completing these simple calculations, she erroneously concluded that because $2^3 < 3^3$, it followed that $e^\pi < \pi^e$. By using crude estimates for $e$ and $\pi$ she trivialized the problem and her arguments led to an incorrect conclusion.

![Figure 59: Amy's work on Problem Set 11, #6.](image)

A review of her submitted problems revealed that Amy experienced greater success with problems requiring geometric reasoning. (Pre-instruction measures had suggested that Amy’s content knowledge in geometry was deficient.) In the following passage, however, her limited understanding of geometric structures impeded her efforts to construct a useful diagram for her solution of the Parallelogram Problem (Figure 60.)

5.) The altitudes of a parallelogram have lengths 8 and 10 and intersect at an angle whose sine is $\frac{1}{4}$. Compute the area of the parallelogram.

![Figure 60: The Parallelogram Problem from Problem Set 13, #5.](image)
Notice how, in her class presentation, she struggled to make sense of the problem, especially the configuration of the altitudes of the parallelogram.

[1] Inst: Okay, what about 5?
[3] Inst: Let's see. So why?
[4] Amy: I didn’t get to finish it. But I’ve almost got it, so —
[5] Jeff: Is there a solution for that one?
[7] Inst: Yeah, I have a solution.
[8] Neo: Can you show it

![Figure 61: Amy's sketch of #5 of Problem Set 13.](image)

[9] Inst: Can I show it?
[10] Ben: Yeah, can you show it?
[11] Amy: Yeah, we're all —
[12] Inst: Ah, I think maybe just this once I will. So, let's see. We know that sine of theta is $\frac{1}{4}$th, all right. So —
[13] Callie: What is the right way to draw the picture?

[14] Amy: I drew a diagonal, which I don’t think you’re supposed to do, and then drew the altitudes

[15] Inst: No. So this is what I have. Turns out my picture is not accurate, but it works out fine for solving the problem. And then I have — <<sketching altitudes>>

![Figure 62: Instructor’s sketch of the same problem.](image)


[17] Amy: So my problem is where is the altitude because I get —

[18] Inst: Well, you can draw the altitude anywhere, right? The altitude is the perpendicular distance between the parallel sides. So I can slide it around, right?

In this episode, it appeared that despite Amy’s attempts to understand the problem and her subsequent reflections of the quality of her solution, her failure in the orienting phase of the problem solving cycle described by Carlson and Bloom (2005) prevented her from successfully planning or executing a solution. Amy's first contribution was the
observation that the solution was incorrect [2] and incomplete [4]. Later in the discussion [14, 18], it became apparent that Amy could not create a useful sketch (Figure 61). The instructor, realizing this difficulty [18], offered a definition of altitude as well as a dynamic image of an altitude in that context (Figure 62).

In summary, Amy’s work on the problem sets submitted during the final phase of the semester provided evidence that she was better able to orient herself to the problem situations and employed many heuristics beyond guess-and-check. She was observed expressing curiosity in and appreciation of solutions that were different than hers. Her scores on these assignments were high, revealing greater persistence and greater problem solving success. However, deficiencies in her conceptual understanding continued to obstruct some of her efforts to find solutions.

Class work

During her interactions with the group, it was observed that Amy continued asking monitoring-type questions in her efforts to make sense of the constructions and computations of others in the group.

In the following episode, Amy pressed Callie to explain herself as they solved the System of Equations Problem (Figure 63).

$$\begin{align*}
\frac{xy}{x+y} &= \frac{1}{2}; \\
\frac{xz}{x+z} &= -\frac{1}{3}; \\
\frac{yz}{y+z} &= \frac{1}{7}
\end{align*}$$

Figure 63: System of Equations Problem

[1] Callie: I have $x$ equals in terms of $y$. I have $y$ equals in terms of $z$. 
Amy: How is that going to help?

Callie: I don't know.

Amy: You have to plug those into —

Then later...

Amy: You're doing just—you’ve got z and y. See if it works right now before we go on.

In this episode, Amy challenged Callie to justify her actions [2] and later, suggested that they verify the work they had completed so far before going further.

Amy continually questioned her group about the Round Trip Problem (Figure 64) as they worked toward a solution.

Suppose that on a round trip, you travel at 30 mph on the way out and at 60 mph on the way back. What was your average speed?

Figure 64: The Round Trip Problem.

The excerpts [1-9] below, taken from a 20 minute episode in the group, illustrate both her role as group monitor and her desire to make sense of the problem and understand its solution.

Amy: Yeah, it was like what do you have for d?

Amy: So basically you do 30 miles, and you use 60 down on the way back. You’re going to do the distance in half the time. Say you do an hour, takes — an hour you do 30 miles. You only do — that in half, an hour and a half, total.

Amy: Why did — why do you —

Amy: — is this distance right here?
[5] Amy: How did you set this up right here? Is that distance times \(rt\)?

[6] Amy: — set this up right here? You have distance but not time?

[7] Amy: How did you set that one up?

[8] Amy: What does this mean?

[9] Amy: I understand it's the total distance. How do you get that?

Excerpt [2] also shows Amy engaging in sense-making before setting up equations and calculating. These questions had the effect of making the group members also reflect on the products and progress of their solution.

While this behavior is encouraging, it is interesting to note that later in the same class session; she reverted relying on numerical calculations and guess-and-check strategies while solving the problem the Blackboard Problem (Figure 65).

A set of consecutive integers beginning with 1 is written on a blackboard. One number is erased. The arithmetic mean of the remaining numbers is \(35\frac{7}{17}\). What number was erased?

Figure 65: The Blackboard Problem.

In the following passage, Amy was observed attempting to determine a numerical interval for the solution by testing different guesses in her calculator.

[1] Amy: So we know it's a number between 70 and 80, but then we got to figure out which number is taken out.

But later she decided that her reasoning might have been faulty.

[2] Amy: But the thing is it can be more than 70 numbers because you can have a sequence of 80 numbers, and then just take a bigger number out to make it —
[3] Inst: Bigger than 80?

[4] Callie: No, but bigger than —

[5] Amy: Okay, so say you have a row of 70 —


[7] Amy: No, and this is why you can't figure it out.


*Continued trying numerous random numbers for about 4 minutes*

[10] Amy: So I think it has to be 70 numbers, so now we can play guess and check.

To solve this problem, an appropriate sub-goal would be to determine how many numbers were in the original sequence. Rather than analyze the problem to find that the sequence must be one more than a multiple of 17, Amy used guess-and-check [1] to set up a solution interval. Her reflection on this idea [2] suggested to her that the problem was unsolvable. The instructor encouraged the group to test her conjecture. After another (incorrect) conjecture in [10], she and Callie went on to “play guess and check” for the remainder of the class without finding the solution. Although this episode was an atypical one during this final phase, it revealed that not only could Amy fall back into unproductive problem solving behaviors, but she also could convince others to join her.

In summary, the class work in sessions 16-22 provided evidence of Amy’s monitoring and sense-making behaviors but also showed her persistent tendency to use guess-and-check as a problem solving tool.
Extended Analysis Tasks

The class completed a two-part extension of the Box Problem during this time.

The first extension of the Box Problem is shown in Figure 66.

Recall the Box Problem solved in class:

Given a sheet of 8.5 x 11 paper, your group wants to construct an open topped box by cutting squares out of the corners. You want to make a box with the greatest possible volume. What are the dimensions of such a box?

Suppose the rectangular paper has length $L$ and width 1 with square cut out with side $x$.

1) Find $V(x,L)$ and then find the $x(L)$ (that is, $x$ in terms of $L$) that produces the maximum volume.

2) Graph $x(L)$ and interpret your results in the context of the problem.

3) Locate the points on the graph that represent the original problem and the square (The 8 ½ x 11 sheet and the 11 x 11 sheet.)

*Figure 66:* The first extension of the Box Problem.

Although Amy was able to construct algebraic representations for the functions $V(x,L)$ and $x(L)$, the fact that she originally set $V$ equal to zero (see Figure 67) suggested that she may not have fully understood these functions.
Figure 67: Amy's work on the first extension of the Box Problem.

She did not interpret the graph of \( x(L) \), as required in the second question and the fact that she graphed both solutions for \( x(L) \) (where one is extraneous in the context of the problem) suggests that she was unable to assign meaning to the graph. Her failure to complete the last part of the assignment supports this conjecture.

The second extension of the Box Problem was intended to compel students to interpret the meaning of the symbols used in the first extension. The figure below (Figure 68) provides the questions asked in this second assignment.
1) If the critical points for the function \( V = x(1-2x)(L-2x) \) are
\[
x(L) = \frac{L+1 \pm \sqrt{L^2 - L + 1}}{6},
\]
determine which is the maximum and which is the minimum. (Recall that \( V \) is a cubic function) Explain your reasoning.

2) What does the graph of \( x(L)_{\text{max}} \) tell us about the problem?

3) Find \( \lim_{L \to \infty} x(L)_{\text{max}} \) and interpret the meaning of that limit.

4) Suppose I have a fixed length of fence that I want to use to create a rectangular shaped pen along the side of a barn. If I call the fixed length “1”, what should the dimensions of the pen be in order to create a pen with maximum area?

5) How does the problem in #4 relate to the Box Problem?

6) Consider the problems we have already solved. (The 8 ½ × 11 sheet and the 11 × 11) and compare the area of the folded up region with the base of the box. Can we generalize the finding to the 1 × L sheet? Convince me one way or the other.

Figure 68: The second extension of the Box Problem.

Amy’s explanation of her choice of the maximizing function (first question in Figure 68) did not reveal how she arrived at her decision (Figure 69), and which of the two functions represents \( x(L)_{\text{max}} \) cannot be determined by inspection alone. Her initial interpretation of the limit (before she crossed it out) reversed \( x \) and \( L \) as dependent and independent variables, suggesting she experienced confusion as to the meaning of those variables. Finally, her substitution of the infinity symbol into the function \( x(L)_{\text{max}} \) as if it were a number might indicate shallow understanding of the concept of infinity.
Figure 69: Amy’s work on the second extension of the Box Problem.

On the other hand, Amy’s solution to the Fencing Problem (Figure 70) (the fourth question in the second extension shown in Figure 68) provides evidence of conjecturing and of reflection upon correctness and validity of her construction.

Figure 70: Amy’s work on the Fencing Problem.
Amy first made the conjecture that the pen would be in the shape of a square. Unlike her behavior in the first segment of the instructional sequence, Amy tested her conjecture. This work also shows that reflections on her constructions led her to find errors in her reasoning, which in turn, led her to abandon that solution path and construct a new, effective, solution path.

When reflecting on the experience, she wrote:

I learned in the Box Problem when we were trying to maximize the area you need to make sure you check the critical points after you find them because one of the solutions for $x$ is not possible and that’s how you determine the maximum between the possible critical points. The part that was difficult was when we had to parameterize the problem and I had to keep track of what they [the parameters] were standing for.

Amy's work on the two assignments associated with the Box Problem revealed that although she had made progress throughout the semester, she still lacked the conceptual understanding of the functions involved required to interpret and understand the parameterized version of the problem.

Final project

As a final project, students were expected to select a high school mathematics problem of their own choosing and independently extend it. Students had the option of working alone or with a partner. Amy, partnered with a classmate, extended the problem shown in Figure 71 for her final project.
A piece of wire is 52 feet long. It is cut into 2 pieces, each of which is then bent into a square. The total area of the 2 squares is 97 square feet. What are the lengths of the sides of the two squares?

Figure 71: The problem selected for Amy's final project.

Once the pair solved the problem and parameterized it, they explored the following avenues:

- Find the minimum area of the two squares when the wire is 52 feet long.
- Find the maximum length of the wire when the area is 97.
- Set the length at 52 feet and vary the area.
- Set the area at 97 square feet and vary the length of the wire.
- Divide the wire into 3 squares.
- Divide the wire into \( n \) squares, \( \{1 \leq n \leq 14\} \).

It appears that their curiosity took them in a variety of directions, allowing them to draw the conclusions below (Figure 72).
Conclusions.

One of the first things that we noticed was that 4 was our magic number. It appeared in patterns throughout our analysis. After playing with the numbers a little, we realized since we were mostly dealing with whole numbers, each side of the square had to be at least 1 unit long, therefore, an entire square consists of 4 units, which is why we see patterns of 4 throughout.

Another thing that we came across, while messing with the minimization problems was that you couldn’t really minimize a rectangle’s area. We had difficulty at first trying to understand what the difference was between a minimization problem and a maximization problem. We realized that we really couldn’t minimize a rectangle’s area because it would just stretch out to where the rectangle’s side would be close to zero.

When dealing with the 3D graphs, it shows us three separate lines that have possible lengths for $x$, $y$, and $z$. The graph is set up almost triangular, showing that the coordinates for $x$ will also be $ys$ and $zs$ on the other two sides of the triangular picture.

When looking at cutting the wire into multiple pieces, we noticed that the maximum areas were decreasing by multiples of 2 so that eventually 13 would be the last possible number of pieces that would still have a length of 52. By looking at this pattern, we didn’t have to finish all of the pieces and could predict when the pattern would end.

As far as the minimums go, when the length was kept constant at 52, 84.5 was the minimum area possible by two squares. The areas went up to 169 which is when there was only 1 square and the other square’s length was zero. When the area was kept constant at 97, 39.39 was the minimum for the length. This length only includes one square, the other square’s length being zero. If there is two squares, one of the lengths is just a bit smaller than 39.39 and the other square’s sides are REALLY small.

*Figure 72: Amy’s statement of conclusions drawn from her final project.*
Amy’s conclusions indicate that through her work on the extension (and with the help of her partner) she was able not only to notice interesting patterns, but she attempted to account for them as well. When compared to earlier extensions (especially the Catching up extension and the two Box Problem extensions), she was able to develop a rich enough understanding of these functions to extend them meaningfully and enrich her mathematical understanding at the same time. During the presentation of this extension, Amy displayed excitement for the interesting aspects of the problem they uncovered.

When asked to reflect upon her experiences with all the extended analyses, she wrote:

> After all our extended analyses as well as our final project, I liked how we were able to look at the problems from different aspects rather than the "normal/traditional" way and most of the times we would conclude something interesting about it. Mathematically I really learned that graphs could give you a lot of different information, so I now like to set up different functions so I can see how they affect the problem. Also, I thought I did well parameterizing the problem and I liked parameterizing the problem because it allowed me to set up a formula for any given value and to see what would happen with a different value. If I could do something different, I would look at more extreme cases because I felt those gave me a little more meaning to the problem.

This reflection is significant in that she stated that she could see advantages to working with a more abstract version of a problem, namely a parameterized version. In light of
Amy’s pre-instruction measures indicating Amy’s utilitarian view of mathematics this is an important shift.

When asked to reflect upon her experience participating in the instructional sequence, she wrote:

> I thought this class was beneficial for us because we went over mathematical topics that are challenging and those problems that have common mistakes made in the high school classroom. I felt going over the different ways the different groups solved the same problem was helpful because it shows that different students have different ways to understand topics and therefore when I’m a teacher I will present different ways that you can approach the same answer and also let the students work together in groups to solve problems. . . . Overall, I found this class to be helpful in preparing me to become a teacher.

Amy’s reflection corroborates observations that she came to value solution strategies and paths that were different than her own. This reflection also suggested that she made a connection between the activities she engaged in as a student and teaching strategies she could adopt as a teacher.

*Summary of sessions 16-22*

Amy’s problem solving abilities and mathematical dispositions continued to evolve during this last phase of the instructional sequence. Analyses of the data revealed that she was better able to orient herself to problem situations and she used a greater variety of heuristics (e.g. graphs, cases, patterns) as she constructed solution paths. She
was observed engaging in sense making, conjecturing and monitoring. An appreciation for abstract representations of situations and for alternative solution strategies emerged during this time period. However, deficiencies in her conceptual understanding continued to obstruct some of her efforts to find solutions.

After Instruction

This section describes Amy’s performance in the post-instruction measures. These measures include the results of a posttest comprised of the same NAEP items as the pretest, a VAMS survey, and a second task-based interview. Additionally, these results will be compared to the pre-instruction measures for evidence of change.

Results from the Posttest.

Amy completed the posttest at the end of the semester. Even though it was identical to the pretest (the complete instrument appears in Appendix A), her responses showed only a modest improvement. Amy answered 24 of the 30 questions correctly, indicating a net gain of just two items. Although she made gains by correctly solving six of the eight items she missed before (Table 8), she also gave incorrect responses on four items that she had correctly answered on the pretest (Table 9) yielding a net gain of 7% on the instrument. Amy’s new errors were somewhat perplexing. The written work associated with #13 suggested that her error was due to carelessness, as she selected the solution which would be correct if the diagram indicated 37%, instead of 37°. Amy indicated on her paper that she did not have a calculator that day, which may or may not account for the other errors.
Table 8

*Amy's Gains on Posttest Items*

<table>
<thead>
<tr>
<th>Item number</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>If triangles ADE and ABC shown in the figure above are similar, what is the value of x?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item number</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>In a group of 1,200 adults, there are 300 vegetarians.</td>
</tr>
<tr>
<td>5</td>
<td>What is the ratio of nonvegetarians to vegetarians in the group?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item number</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>8</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>In the figures above, the radius and height of each right circular cylinder are given. If w, x, and y represent the respective volumes of the cylinders, which of the following statements is true?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item number</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>The least common multiple of 8, 12, and a third number is 120. Which of the following could be the third number?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item number</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td><img src="image4" alt="Display" /></td>
</tr>
<tr>
<td></td>
<td>The figure above shows the display on a scientific calculator. The value of the displayed number is between which of the following pairs of numbers?</td>
</tr>
</tbody>
</table>
Semicircles are constructed on the sides of an equilateral triangle, as shown in the figure above. Of the following, which best approximates the sum of the lengths of the three darkened arcs?

\[ n=6 \]

\[ \text{Table 9} \]

**Amy's Incorrect Posttest Items**

<table>
<thead>
<tr>
<th>Item number</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>27</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>Semicircles are constructed on the sides of an equilateral triangle, as shown in the figure above. Of the following, which best approximates the sum of the lengths of the three darkened arcs?</td>
</tr>
<tr>
<td>1</td>
<td>A circle with diameter 10 centimeters is to be cut from a square of paper 10 centimeters on a side. Of the following, which is closest to the amount of paper left over after the circle is cut out?</td>
</tr>
<tr>
<td>3</td>
<td>( \cos^2(3x) + \sin^2(3x) = )</td>
</tr>
<tr>
<td>13</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td></td>
<td>The entire circle shown above represents a total of 2,675 radios sold. Of the following, which is the best approximation of the number of radios represented by the shaded sector of the circle?</td>
</tr>
<tr>
<td>26</td>
<td>The postal rate is 25 cents for the first ounce and 20 cents for each additional ounce or part of an ounce. What would it cost to mail a package that weighs 6.8 ounces?</td>
</tr>
<tr>
<td></td>
<td>[ n=4 ]</td>
</tr>
</tbody>
</table>
Table 10

Amy’s Gains by Content Area

<table>
<thead>
<tr>
<th>Content area</th>
<th>Number of items</th>
<th>Number of correct items on pretest</th>
<th>Number of correct items on posttest</th>
<th>Percent change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry and Spatial Sense</td>
<td>n=4</td>
<td>3</td>
<td>4</td>
<td>+25%</td>
</tr>
<tr>
<td>Number sense, properties, and operations</td>
<td>n=8</td>
<td>5</td>
<td>7</td>
<td>+25%</td>
</tr>
<tr>
<td>Measurement</td>
<td>n=4</td>
<td>1</td>
<td>2</td>
<td>+25%</td>
</tr>
<tr>
<td>Algebra and functions</td>
<td>n=10</td>
<td>9</td>
<td>8</td>
<td>-10%</td>
</tr>
<tr>
<td>Data analysis, statistics, and probability</td>
<td>n=4</td>
<td>4</td>
<td>3</td>
<td>-25%</td>
</tr>
<tr>
<td>30 total</td>
<td></td>
<td>22</td>
<td>24</td>
<td>+7%</td>
</tr>
</tbody>
</table>

Amy’s responses, organized according to content area (Table 10), reveal gains in items from the Geometry and Special Sense, Number Sense Properties and Operations and Measurement categories. These gains are not surprising as these were the areas revealed as deficient in the pretest. When clustered by ability type (Table 11), Amy made a very modest gain in items categorized as Procedural Knowledge (Table 10), suggesting that she was better able to match concepts and procedures. It is very interesting that Amy evidenced no gains in the Problem Solving ability type.
Table 11

Amy's Gains by Ability Type

<table>
<thead>
<tr>
<th>Mathematical Ability</th>
<th>Number of items</th>
<th>Number of correct items on pretest</th>
<th>Number of correct items on posttest</th>
<th>Percent change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem Solving</td>
<td>13</td>
<td>10</td>
<td>10</td>
<td>+0%</td>
</tr>
<tr>
<td>Conceptual Understanding</td>
<td>10</td>
<td>8</td>
<td>9</td>
<td>+10%</td>
</tr>
<tr>
<td>Procedural Understanding</td>
<td>7</td>
<td>4</td>
<td>5</td>
<td>+14%</td>
</tr>
</tbody>
</table>

|                       | 30              | 22                                | 24                                  | +7% overall    |

Amy's performance on the posttest showed an overall gain of 7%, as her score rose from 22 to 24 correct of 30 questions. These data indicate although Amy was able to correctly answer 80% of the items on the instrument, her score was only slightly higher than her pretest score.

Views About Mathematics Results

The results of Amy’s post-instruction VAMS revealed positive shifts toward expert views about the practice of mathematics. Even though Amy’s pre-instruction VAMS categorized her as holding naïve views, her post-instruction VAMS categorized her as holding expert views. Comparison of the number of responses where Amy selected expert, mixed and naïve options (Table 12) suggests that her beliefs about mathematics had shifted.
Table 12

*Comparison of Amy’s Pre- and Post- VAMS Results*

<table>
<thead>
<tr>
<th>Survey</th>
<th>Number of expert responses</th>
<th>Number of mixed responses</th>
<th>Number of folk responses</th>
<th>View category</th>
</tr>
</thead>
<tbody>
<tr>
<td>pre</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>Naïve</td>
</tr>
<tr>
<td>post</td>
<td>14</td>
<td>6</td>
<td>1</td>
<td>Expert</td>
</tr>
</tbody>
</table>

Items where Amy demonstrated the greatest shifts (Table 13) were VAMS items 4, 5, 10, 19, 20 and 24.
Table 13

*Items Indicating the Greatest Gains on VAMS*

<table>
<thead>
<tr>
<th>VAMS items showing change</th>
<th>Amy’s pretest</th>
<th>Amy’s posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
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<tr>
<td>4</td>
<td></td>
<td></td>
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<tr>
<td>5</td>
<td></td>
<td></td>
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<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Towards “Only (a)”

Equally (a) & (b)

Towards “Only (b)”

4. My score on mathematics exams is a measure of how well:
   
   (a) I understand the covered material.
   
   (b) I can do things the way they are done by the teacher or in some course materials.

   [Not categorized by Carlson et al.]

5. For me, doing well in mathematics courses depends on:
   
   (a) how much effort I put into studying.
   
   (b) how well the teacher explains things in class.

   {Expert View: options 1-4; Mixed View: option 5; Folk View: options 6-7}

10. Mathematical formulas:

    (a) express meaningful relationships among variables.

    (b) provide ways to get numerical answers to problems.

    {Expert View: options 1-3; Mixed View: option 4; Folk View: options 5-7}
19. Collecting and graphing real world data is useful for:

(a) determining patterns and making general predictions.

(b) obtaining numerical answers to specific problems.

20. For me, making unsuccessful attempts when solving a mathematics problem is:

(a) a natural part of my pursuit of a solution to the problem.

(b) an indication of my incompetence in mathematics.

24. In order to prove a mathematical theorem one must:

(a) produce evidence from the physical world.

(b) provide a logically sound argument.

Amy’s VAMS responses to #10, #19, and #24 suggest a shift towards a view that the study of mathematics is more about patterns and relationships and less about ways to find numerical solutions to problems. Her response to #20 suggests that she might be
developing a more mature view of herself as a problem solver by acknowledging that unsuccessful solution attempts are part of the problem solving process. Items #4 and #5 are interesting because they deal with the roles and responsibilities of teachers and students in the class. The movement in these items reveals that for Amy, successful teaching and learning of mathematics may no longer be just about how well a teacher explains things or how well a student replicates a teacher’s work. This shift is interesting because these ideas were never formally addressed during the course, suggesting that the experience of the way the course was conducted may account for the change. It is worth noting that Ben demonstrated shifts on the exact same survey items, suggesting that these may be attitudes that were fostered by the instructional sequence.

The Post-Instruction Interview

The next pages report the results of a second semi-structured interview with Amy conducted at the conclusion of the instructional sequence. The interview lasted approximately 60 minutes and was conducted by me (the researcher and instructor). Amy was asked a variety of questions about her experiences during the class (refer to Figure 8), and was asked to solve two mathematics problems (refer to Figure 9) using a talk aloud protocol.

Responses to the general questions

When asked if she felt her knowledge of mathematics had grown or changed as a result of participating in the course, Amy stated:

It’s both grown and changed in the ways that when I’m given a problem, where I look at it, I seriously, I never used graphs before and every time we did a problem in a group, I would try to do the algebraic way where
someone else would do it graphically to find the same answer, so I really liked how you could do that, so now I try to see if I can set something up to look at the graph as well to see what it gives you. Because sometimes it will give you answers like for not always the one you’re working. Like, it will give you that one but also will give you further data.

In this passage, it appears that Amy recognized the shortcomings of the strictly algebraic approaches to problem solving that she exhibited early in the semester and now she saw advantages to more abstract representations and thinking. In the next passage, she acknowledged that problems involving rates continued to be difficult for her.

The stuff we did with rates, I totally forgot all my Calculus; I didn’t really, it’s not that I didn’t pay attention, but I didn’t have good teachers I think. But Calc III had a teacher that actually explained it to me, but I haven’t seen it since the beginning of sophomore year, so to go over that stuff because I don’t have the best knowledge on that. Like, those are always tougher problems to think about.

When Amy was asked if she noticed changes in her own problem solving abilities, she noticed that she had become more persistent. "In the fact that I might not give up, but sit down more and maybe try and reason more." As mentioned in the previous section, Amy’s improved persistence was evidenced by the fact that she submitted more completed assignments as she progressed through the instructional sequence. Her response to VAMS item 20 also suggested that Amy had become more persistent.
Amy credited productive group work (which she had not experienced in previous mathematics classes) as contributing to improvements in her problem solving abilities.

I liked it [group work] because of the different approaches. And then also, I mean the graph stuff was important but then also to do extended analyses I found writing a formula for stuff was good, so we’d always write up a formula together in order to help us solve something and then — what other two types, there’s — I learned this in my Methods class that — Dora says this all the time — process view and action view. I’m action not process though. I don’t normally set up a formula that will work for all times but- I mean I see the formulas, which are great, but I would still probably do it individually.

I think you typically approach problems certain ways so in my group was Callie and Ben. Ben was always – Callie actually was too – formula, like she did formulas all the time, and then Ben would just look at the problem, sometimes like different ways, that I wouldn’t typically think of. And I think I had, I don’t know if it’s a more standard approach, but I like more algebra.

This statement supports earlier assertions that Amy was coming to appreciate and value solution strategies and paths that differed from hers, as does the following statement:

I did like when we worked on problems in our group in class because we were able to see how other people approached it so that we would think about when we become a teacher, I mean, you can explain it various types of ways. I mean, because there are different types of teaching because
there are different types of learning styles so that all the students can understand. I liked that because I would do one thing, someone would do something else. And like, “So how did you do this?” and try and follow their work and they would try and explain it to me.

The Extended Analysis Tasks also appeared to be a useful tool for supporting Amy’s shift away from the calculation-based reasoning that was so prevalent in her early work.

I liked them (Extended Analysis Tasks) at first, I just need to learn how, like what was being asked of me. I didn’t know what they meant by “extreme cases” and when someone gave a presentation on one, I was like, “Oh, that’s how it’s supposed to be done.” I think it [seeing presentations] was helpful because I was like, how are we supposed to look at this in the “extreme case,” what’s different?

For me, it [what I learned was] wasn’t that one case, it basically was the process view to show you for all cases where I thought it was meant to show us because we, what is it called? Parameterized.

In summary, Amy’s responses to these questions revealed that she believed that her understanding of mathematics had been enriched by her experience in the class and that she experienced improvement in her problem solving abilities. According to Amy, Extended Analysis Tasks provided an appreciation for more abstract mathematical representations. She also found that productive group work and exposure to multiple solution paths were especially beneficial to her.
*The problem solving session*

Amy was asked to attempt two problems and to share her thinking aloud. The first problem she attempted was the Quarter-circle Problem (Figure 73).

The quarter circle shown above has center C and radius 10. If the perimeter of rectangle CPQR is 26, what is the perimeter of the shades region?

*Figure 73: The Quarter-circle Problem.*

In the following excerpt, Amy showed evidence of orienting herself to the problem by labeling her diagram (Figure 74) and assigning variables. Then she verbalized possible strategies and reflected on them.

[1] Amy: Um... let’s see. The triangle within the rectangle is $x^2+y^2$ will equal $z^2$. So I can figure out what that one is to add up the perimeter.

[2] Amy: I’m going to have a ton of variables. AP is 10-x because it’s the length.

[3] Amy: Same thing here, BR=10-y. Now what do I need to do? This is 90. This arc length from A to B is, I’m trying to think, that’s different. Oh, well, hold on hold I’ve got to think. It’s probably an easy thing; the fact that this is 90 I’ve got to think about it. Umm, what would the whole circle be if the area plus the radius is 10.
Figure 74: Amy’s work on the Quarter-circle Problem.

[4] Amy: Ok. Circumference is $2\pi r$ which will be the whole circle and I’ll just take a quarter of that. Umm, circumference equals — not 10, $20\pi$ and then if we want to know this section, I’m going to multiply by .25 I believe, not .75. This is 25, a quarter of that would be, or you subtract .75 of the $20\pi$…

[5] Amy: Because if the radius is 10 and then I’m going to subtract .75 times $20\pi$ to give me the length of $AB$. This equals $AB$.


[7] Amy: Now I have fun with variables, let’s see. I’ll set this one, no that’s not going to help me; I’ll have to set something else up.

$y=12-x$. 
Amy: Oh, I see something possible: I know that length \( AC \) should equal \( CB \) so what I’m going to do is \( 10-x+x \) should be, well 10 equals 10. Still working \( x^2=13-x \). \( 13-x^2 \) will equal \( C^2 \) but, ok…

Int: So what are you thinking?

Amy: Well in this case right here, I have two unknowns. I have, using this I have two equations but if I include \( z \), I have three unknowns. And you need more information to find out that last variable.

Amy: What else do I have to use? I have this length to find these lengths I need to find out what \( x \) and \( y \) is. I won’t keep going with this, I’m not seeing anything.

In [1], she labeled her diagram and set an appropriate sub-goal (finding the length of \( z \)). In [3-5], Amy appeared to be considering various solution strategies. She was also observed reflecting on the various pieces of information she needed to solve the problem throughout the passage by engaging in self talk such as “what do I need to do?” in [3] and “no, that’s not going to help me” in [7] which appeared to have productive results. In this example, Amy demonstrated orienting behavior such as organizing the information and making appropriate constructions. She also set sub-goals, and verbalized a problem solving strategy. As mentioned earlier, Amy’s pre-instruction problem solving session revealed little evidence of orienting or planning. It is also encouraging to note that Amy did not resort to numerical calculations to gain traction with the problem as she had in the past. Additionally, in this passage, her monitoring appeared to help her change direction and avoid what Schoenfeld (1987b) called “wild goose chases”.

Although Amy abandoned this problem when offered the opportunity (as she had in the pre-instruction problem solving session), she was willing to revisit the problem later in the session. In the following passage, even though she was unable to solve the problem, she could see a path to the solution.

[1] Int: Just a thought. Do you want to go back and look at that first problem again?

[2] Amy: Yeah. What piece am I missing? Basically here I can make x in terms of y and y in terms of x. This is my other equation; I need to maybe find another one. And…

[3] Int: So you’re saying that z is like the thing if you could find out z was, you could find everything else.


In [2], Amy articulated what was required to complete the solution. Amy worked another five minutes without success. The central obstacle appeared to be her limited conceptions of the geometric structures in the problem. In particular, Amy failed to access the fact that the length she called z had the same length as the radius of the circle. This passage corroborated earlier assertions that Amy’s mathematical understandings impeded her efforts to construct viable solutions.

The second problem (Figure 75) offered Amy an opportunity to grapple with the concept of rate. Earlier (when responding the general questions), she had admitted that she continued to struggle with her conceptions of rate. In the episode that follows, Amy was able construct a graph after some initial confusion about dependent and independent variables.
Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that is in the bottle.

Figure 75: The Bottle Problem.

[1] Amy: Ok. Well, first of all, height as a function of the water. Ok. First, I’m going to write height here, amount and water over there but then I’ll check my work. Height as a function — of height — function. I’m not sure if I have my axis labeled correctly. This is just word order. Height as a function of water: it might be opposite.

[2] Int: How did you decide?

[3] Amy: I think I have it backwards, but I wasn’t sure. Because I see it as a function of water, which is \( f(x) \) where this is \( x \)... I’m still not 100% sure. So I’ll make this height. Ok. And I understand this question.

[4] Amy: Ok so as this is filled up the height is going to get, let’s see. Ok, as you add more water and the height is clearly always going to get taller but at different rates because of the way this is bulging out. So, here, here, here, fill up to here, the changing. Ok. This isn’t half, let’s see, what should we call it? I’m still calling it “half full,” \( \frac{3}{4} \). Ok. Basically as we fill this space with water, the height will slowly get higher but then from \( \frac{1}{2} \) to \( \frac{3}{4} \) it will rapidly get...
higher because it’s filling up a smaller area. ¾ to 1, it’s going to fill up at, well, a constant rate. I imagine that one’s slope to being right. I think ¾ to 1 might just be a $y=x$ slope. Because it’s going to get higher as it gets filled, but it, won’t it be constant like this because that means it’s not growing. So I’m thinking this is like $y=x$ between ¾ and 1.

[Int: Ok.]

[Amy: Just, I mean I know it’s going at the same rate, but it’s getting larger so I think that’s how you might show that. This one went from ½ to ¾ would go up quickly as well. This one’s slower, so how can we do this?]
here to here it will only get that high, but from here it’s going to
double in height, that’s sort of, so do take this part of the drawing
in terms of how far apart they are. It would double, and this one’s
even smaller so it’s going to even double, so \( x_3 \)’s up here probably.

Are you following what I’m doing?

Amy’s attempts to make sense of the problem statement are revealed in lines [1-3]. In
lines [4-7], she carefully considered the rate at which the height of the water in the bottle
would change and its impact on the shape of the graph, including the slope of the part
representing the cylindrical part of the bottle. Her first attempt at graphing the function is
shown in Figure 76. When asked if the graph represented her final answer [8], she
revisited her thinking.

[8]  Int: Yeah. So is that your final answer?


[11]  Amy: You’re pouring water half way; it’s the bottom part I have to
    check. I’m pretty confident about the rest. Here the height, because
    the height from here to here is going to be quicker than the height
    from here to here.


[13]  Amy: Seeing if I represented it correctly in my graph because if I
    make the slope any larger, it will seem like it’s doing a lot but then
    I’ll have that because it’s going to be slower but then here it will be
greater and greater. So …
Later, after creating a second sketch

Figure 77: Amy's final solution to the Bottle Problem.

[14] Int: Ok. That’s your final answer?


Amy’s statement in [9] indicated that she still needed to check her solution. In [11] and [13] she appeared to be evaluating her construction. Her reflection on her work product alerted her to the fact that she had not represented the "bottom part" of the bottle correctly on the graph, and created a second sketch (Figure 77). When asked again if she had a final solution [14] Amy appeared to be confident of her work [15].

As Amy solved the Bottle Problem, it appeared that monitoring behaviors such as self talk and reflecting on her work products prompted her to refine her sketch, thus producing a more accurate representation. Her reflection on her thinking convinced her of the validity of her solution, while in the pre-instruction interview; she tended to look to
the interviewer for verification. This suggests that Amy had more confidence in her own mathematical judgment than she had prior to instruction.

Amy's problem solving session during the post-instruction interview revealed that she was able to orient herself to the problem space, set appropriate sub-goals, and verbalize solution strategies. She did not rely on numerical calculations or on guess-and-check strategies that were so prevalent in the pre-instruction problem solving session and during the first segment of the instructional sequence. She used self-talk as an effective means to monitor her work. Despite these improvements, her conceptual understanding of mathematics prevented her from solving the first problem.

These data together with the other post-instruction measures (posttest, VAMS, interview) suggest that Amy experienced shifts in both her problem solving behaviors and her beliefs about mathematics and mathematics teaching even though deficiencies in her content knowledge at times derailed her problem solving efforts.

Summary

In Amy’s pre-instruction interview, she stated that she believed she was a good problem solver and that she felt that she was persistent in her efforts to solve difficult problems. She also expressed confidence in her knowledge of high school mathematics. Yet, her pre-instruction measures suggested that her conceptual understanding of high school mathematics (conceptions of rate, geometric reasoning and measurement in particular) were not well developed. VAMS results suggested that she held naïve views about mathematics. Amy’s problem solving session suggested that she began the execution phase of the problem solving cycle without fully understanding the problem statement or verbalizing a plan. She tended to look to the interviewer for guidance and
verification (as opposed to her own logical constructions), and she appeared to have few tools at her disposal when she hit dead ends. In short, her problem solving session revealed deficiencies in all areas of the Multidimensional Problem Solving Framework. Her performance on all three of the measures (pretest, VAMS, interview) indicated that she is one of the weakest students in the class.

During this early phase of the course, Amy's problem solving behavior was similar to that revealed in the pre-instruction data. She experienced difficulty accessing appropriate mathematical knowledge when attempting to solve problems. Her impoverished and poorly connected knowledge base impeded her ability to make sense of problems, construct viable solution paths, and overcome failed efforts. She used few heuristics and often resorted to numerical calculations to both make sense of and find solutions for problems. These deficiencies led to her being easily frustrated and giving up on problems. In addition, there was little evidence that she monitored her work or her thinking during this segment of the instructional sequence.

However, starting with the sixth class session, changes in Amy’s mathematical behavior became apparent. Her assignments revealed improvement in her ability to construct productive solution paths and valid mathematical arguments. Amy was observed using heuristics other than guess-and-check, including examining cases and creating sense-making diagrams. It also appeared that Amy was more persistent, as evidenced by fewer unfinished homework problems. In her group interactions, she was observed working hard to both follow and reflect on the work of her peers. Through her work on the Triangle Problem, she expressed the realization that she needed to expend more effort in the orienting and planning phases of problem solving. Amy was able to
construct isomorphic problems through her work on the Extended Analysis Tasks, and showed excitement and curiosity in her efforts to extend the problems. Although Amy’s conceptual understanding of the mathematics at hand continued to be her greatest obstacle, during this phase, she appeared to be more motivated to enrich her mathematical conceptions.

During the last segment of the instructional sequence, Amy’s problem solving abilities and mathematical dispositions continued to evolve. She was better able to orient herself to problem situations and she used a greater variety of heuristics (e.g. graphs, cases, patterns) as she constructed solution paths. She was also observed engaging in sense making, conjecturing and monitoring. Her appreciation for abstract representations of situations and for alternative solution strategies also emerged during this time period.

Her post instruction measures revealed that although she made only modest gains on the content-focused posttest, she made measurable shifts in her attitudes away from viewing mathematics as a body of knowledge about numbers and computations towards viewing mathematics as a body of knowledge about ideas and relationships. Amy reported that the course had positively influenced her understanding of mathematics. As well, it had helped her to experience instruction that was consistent with the instruction she hoped to provide for her students. She reported that the Extended Analysis Tasks, group work and presentations of multiple solution paths were valuable experiences for extending her problem solving abilities.

The problem solving sessions revealed that even though conceptual understanding continued to obstruct her efforts to perform mathematical tasks, she exhibited observable
progress in adopting the desired behaviors and dispositions described in the MPS Framework. Although quantitative measures did not reveal significant gains in conceptual understanding, the data support that Amy's problem solving behaviors and her thinking about mathematics in general had undergone an evolution.
CHAPTER SEVEN
THE CASE OF BEN

This chapter presents results from analyzing data that was collected from Ben. Results from analysis of Ben’s problem solving sessions are presented to characterize shifts in Ben’s problem solving abilities as he progressed through 22 sessions of the instructional sequence. Data is also presented to characterize Ben’s beliefs about mathematics, confidence in his mathematical abilities and persistence in working through complex problems. The Multidimensional Problem-Solving (MPS) Framework discussed in detail in Chapter Three was used to analyze this data and characterize these shifts. In particular, evidence of orienting, planning, executing, and checking are provided and discussed.

The first section characterizes Ben’s mathematical dispositions prior to the instructional sequence. Results from the content-focused pretest (Appendix A) provides information about his mathematical abilities with respect to high school content. Results from Ben’s completion of the Views About Mathematics Survey (VAMS) (Appendix B) are presented. Analysis of his pre-instruction interview are presented; this data provided information about Ben’s previous mathematical experiences and problem solving abilities.

The second section presents data from analyzing Ben’s written solutions to class assignments. It also included data from analyzing Ben’s responses to in class tasks that were completed in collaboration with his peers. Results from analyzing Ben’s responses
to the five extended analysis tasks that were completed during the semester are also included.

The third section presents data from Ben’s performance on post-instruction measures. Shifts in Ben’s performance are reported by comparing pre and post instruction data.

The chapter concludes with a summary of insights gained from investigating Ben’s problem solving behavior.

**Prior to the Instructional Sequence**

Ben was a junior at the university when he enrolled in the experimental class, and he anticipated that he would be student teaching within the year. He had completed the three-course Calculus sequence as well as courses in Linear Algebra, College Geometry, Mathematical Structures, Discrete Mathematics, Differential Equations, and Methods for Teaching Mathematics, and was enrolled in Number Theory. He had also either had completed or was concurrently enrolled in courses required by the College of Education for a teaching certificate. Ben had completed more upper division mathematics courses than Amy at the time of the study.

**Results from Pretest**

Ben completed a pretest consisting of 30 items chosen from 2003 National Assessment of Educational Progress (NAEP) grade 12 items (see Appendix A for the pre and posttest document). All items were multiple choice and all were categorized as hard according to NAEP documents. Ben answered 27 out of 30 questions correctly on the pretest; Table 14 shows the three items that Ben answered incorrectly. It is interesting that two of the three items (#6 and 16) were also answered incorrectly by Amy. His
performance on this measure revealed that he was able to provide answers to high school mathematics problems. When compared to his classmates, Ben’s score ranked near the top (refer to Appendix A for scores of other members of the class).

Table 14

Ben's Incorrect Responses on the Pretest

<table>
<thead>
<tr>
<th>Item number</th>
<th>Pretest item</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>A rectangular pool 24 feet long, 8 feet wide, and 4 feet deep is filled with water. Water is leaking from the pool at the rate of 0.4 cubic foot per minute. At this rate, how many hours will it take for the water level to drop 1 foot?</td>
</tr>
<tr>
<td>16</td>
<td>A certain company keeps a list of 50 employees and their annual salaries. When the salary of the very highly paid president is added to this list, which of the following statistics is most likely to be approximately the same or nearly the same for the original list and the new list?</td>
</tr>
<tr>
<td>18</td>
<td>In the figures above, the radius and height of each right circular cylinder are given. If w, x, and y represent the respective volumes of the cylinders, which of the following statements is true?</td>
</tr>
</tbody>
</table>
Analysis of Ben’s performance in each of the five NAEP Content Areas\(^5\) (Table 15) revealed that he answered half of the questions categorized in the content area of Measurement incorrectly (#6, #18), suggesting that this was an area of difficulty for him.

**Table 15**

*Ben's Errors by Content Area*

<table>
<thead>
<tr>
<th>NAEP Content area</th>
<th>Items missed by Ben</th>
<th>Number of items</th>
<th>Percentage of items missed by Ben</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry and Spatial Sense</td>
<td>all correct</td>
<td>n=4</td>
<td>0%</td>
</tr>
<tr>
<td>Number sense, properties, and operations</td>
<td>all correct</td>
<td>n=8</td>
<td>0%</td>
</tr>
<tr>
<td>Measurement</td>
<td>#6 and #18</td>
<td>n=4</td>
<td>50%</td>
</tr>
<tr>
<td>Algebra and functions</td>
<td>all correct</td>
<td>n=10</td>
<td>0%</td>
</tr>
<tr>
<td>Data analysis, statistics, and probability</td>
<td>#16</td>
<td>n=4</td>
<td>25%</td>
</tr>
</tbody>
</table>

\(^5\) See Appendix A for information about the definitions of these content areas
Table 16 shows Ben's pretest results clustered by NAEP Mathematical Ability\(^6\).
The data suggest that he could successfully answer problems that were categorized in all
three ability types. In particular, his results in the area of *Procedural Knowledge*, which
NAEP defined as a student's "ability to connect an algorithmic process with a given
problem situation, to employ that algorithm correctly, and to communicate the results of
the algorithm in the context of the problem setting" (National Center for Educational
Statistics, 2001), suggest Ben was able to access the appropriate mathematical resources
required for these problems.

Table 16

*Ben's Errors by Ability Type*

<table>
<thead>
<tr>
<th>Mathematical Ability</th>
<th>Items missed by Ben in ability type</th>
<th>Number of items in pretest</th>
<th>Percentage of items missed in pretest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem Solving</td>
<td>#6 and #18</td>
<td>n=13</td>
<td>15%</td>
</tr>
<tr>
<td>Conceptual</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Understanding</td>
<td>#16</td>
<td>n=10</td>
<td>10%</td>
</tr>
<tr>
<td>Procedural Understanding</td>
<td>all correct</td>
<td>n=7</td>
<td>0%</td>
</tr>
</tbody>
</table>

\(^6\) See Appendix A for information about the definitions of these Ability Types
Analysis of Ben's responses to the pretest items suggests that Ben’s mathematical background is well-rounded with respect to the abilities assessed by the NAEP items. His responses in the content area of Measurement suggest that there may be some deficiencies in his knowledge with respect to the items in this category.

Views About Mathematics Results

Ben completed a Views About Mathematics Survey (VAMS) at the beginning of the course. Ben’s responses were then categorized as expressing expert, mixed, or folk views in accordance with the classification scheme set out by Carlson, et al. (1998). Once the responses were classified, the number of responses in each category was counted to determine if the survey indicated someone espousing expert, upper transitional, lower transitional, or naïve views about the practice, teaching, and learning of mathematics. Analysis of Ben’s survey revealed 13 expert responses, 6 mixed view responses, and 2 naïve responses, placing him in the expert classification.

Ben's relatively high number of responses classified as expert suggest that he held views about knowing and learning mathematics that are compatible with views commonly held in the mathematical community. That is, his selection of the items where he chose expert options suggest that he viewed mathematics as knowledge of ideas and relationships, and that teaching and learning mathematics entails understanding mathematical concepts as opposed to merely performing calculations.

Table 17 contains the two questions revealing Ben’s naïve views about mathematics. His response on item #17 suggests some uncertainty about the use of mathematics in modeling. His response on #20 suggests that he saw unproductive solution paths as reflecting his lack of knowledge. It is interesting to note that the folk
options on these two particular items were selected by the majority of students in the class.

Table 17

*Items in Which Ben Demonstrated "Folk Views" About Mathematics*

<table>
<thead>
<tr>
<th>Item where Ben responded with a &quot;Folk View&quot;</th>
<th>Ben's response</th>
</tr>
</thead>
<tbody>
<tr>
<td>17. When they represent relationships in the physical world, mathematical functions are:</td>
<td>4</td>
</tr>
<tr>
<td>(a) exact expressions of what is being represented.</td>
<td>{Expert View: options 6-7; Mixed View: option 5; Folk View: options 1-4}</td>
</tr>
<tr>
<td>(b) approximate expressions of what is being represented.</td>
<td></td>
</tr>
<tr>
<td>20. For me, making unsuccessful attempts when solving a mathematics problem is:</td>
<td>4</td>
</tr>
<tr>
<td>(a) a natural part of my pursuit of a solution to the problem.</td>
<td>{Expert View: options 1-2; Mixed View: option 3; Folk View: options 4-7}</td>
</tr>
<tr>
<td>(b) an indication of my incompetence in mathematics.</td>
<td></td>
</tr>
</tbody>
</table>

Overall, results from VAMS reveal that Ben already held the views that the instructional design intended to promote, namely that mathematics is a coherent body of
knowledge about patterns and relationships that makes use of systematic methods and is validated by logical proofs (Carlson et al., 1998).

Pre-Instruction Interview

The next pages report the results of an interview with Ben during the first week of the semester. The interview, held immediately after class, lasted approximately 60 minutes and was conducted by me (the researcher and instructor). Ben was asked a variety of questions about his previous mathematical experiences (refer to Figure 6) and was asked to solve three mathematics problems (refer to Figure 7) using a talk aloud protocol.

Responses to the general questions

When asked, "What kind of math are you good at," Ben responded with the following:

Well, I'm pretty good at geometrical-type stuff, I'm pretty good at using angles and theorems and properties to prove the different geometrical shapes and angles and stuff. I did really well in Math 300, which was good, so I like doing proofs just because, even though they might be ambiguous and stuff, usually they're usually pretty cut and dried. And problem solving – usually I can come up with some creative ways, if not the usual way, to solve the problem.

The passage reveals that Ben saw himself as logical and comfortable with abstract thinking. These statements are closely aligned with the pretest results where he performed well on items assessing Geometry and Spatial Reasoning as well as items
categorized as requiring problem solving and conceptual knowledge. The excerpt that follows characterizes his conceptions about problem solving.

[1] Int: When I say problem solving, what comes to mind?

[2] Ben: When I hear the word *problem solving* I guess I generally think of a word problem. I don't know, just like regular problem solving?

[3] Int: Yeah


[5] Int: Would you include "proving" as a kind of problem solving?

[6] Ben: I think so. I think if you, I mean, if you, I mean that could be a problem in itself, to figure out if, like, for instance, when I did that thing where sometimes you prove by induction that the induction type thing actually equals like some $n^2$ or whatever, like to know how you have to prove it, so that could be a problem so yeah, that would be proving.

[7] Int: Okay. How would you describe your problem solving abilities then?

[8] Ben: Pretty good. Pretty good. Just being able to, I don't know, the first part is just looking at the whole thing and getting the big picture, because sometimes when you have like a long proof you forget exactly what you are trying to prove and so if you can see exactly what's going on and what you need to do it's a lot clearer and you can come up with more — I don't know, more insight as to what you have to do, to solve that proof.

In [4], Ben indicated that for him, problem solving is a creative venture, and in [6] he revealed that his conception of problem solving was broad enough to include proof construction. He also stated his confidence in his abilities in [8]. In the following, he
discusses his level of confidence with the topics of high school mathematics in more detail.

[I feel] pretty confident. I think the one thing that I struggle with that I still — well I took calculus in high school, I took the AP test and I took calc. again there just to make sure I was good to go, and I think the one thing I continued to struggle with was the notation of, like, using derivatives, like for $d$, like substituting, switching $dx$ and this went from all the way up to Calc. 3, like using the $\frac{dx}{dt}$ and all that with chain rule and stuff, like interchanging those, always gave me problems, so I would have to, would have to brush up on that. I know high school calculus would be pretty easy. And then another, I don't think I've ever properly explained how to get from the summation to the integral uh, it makes sense now, but back then it was always something [I didn’t understand].

Like Amy, he expressed confidence in his knowledge of the mathematics he would teach. However, in the preceding passage he revealed insight into gaps in his conceptual knowledge, particularly in Calculus. It is interesting that he expressed an interest in teaching Calculus to high school students.

I think I would want to teach calculus. I think, well, it's always cool to teach someone something new, you know. So either way it's going to be fun. But I kind of enjoy, like higher level thinking, a little bit higher level thinking, and once you get into calculus there's lots of different ways you can do it, instead of like algebra and like linear equations, there's really
only one way to solve them, you know, and then once you get higher and higher, there's more and more ways so that the kids start to pick up patterns and I think that, that'd be more fun. Then also, one of the biggest things that I saw which was a huge problem at Tempe High was understanding like different graphs, like what they look like, like cubed and squared, and once you start to understand what they look like, and the transformations, it makes like college a lot easier.

In the next passage, he expressed his experience with frustration and persistence during problem solving.

[1] Int: So, how do you manage frustration when you're working on problem solving, doing math homework and stuff?

[2] Ben: Well typically when I do math homework I get frustrated, I think, like what don't I, am I missing something, something in class, that's not helping me to do, typically a math problem or problems in math, the things you've done, the things you have do in the problems, you just need exactly what you've just gone over or previous thoughts of "I'm not getting it" or review the lesson or even way further back, and write it down and ask for further help, class notes, the proofs that have to be in a book. When I get stuck on them it's because I don't, I don't remember a property that was given like a chapter before, you know? So just review everything I've learned, if I'm still stuck, sometimes I'll ask for a different view from somebody else, or try to explain it to somebody else, and that sometimes helps but —
Ben reported having developed tools and strategies for managing his frustration [2] and stated that he is able to persist with difficult problems [3-12]. These statements are aligned with the expert views revealed in his responses to the Views About Mathematics Survey (VAMS). While Amy stated that she tended to go searching for a formula or

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7 See Figure 95 for the Dartboard problem.
procedure to help her, Ben’s comments suggest that he was likely to review notes and examples in an effort to understand the problem or situation better. His experience carrying problems around suggests that he could be persistent in his efforts to solve difficult problems. In the next passage, Ben also expressed that planning a solution might be helpful, although his tendency was to pursue the first solution path that occurred to him.

Kind of start right in, and then move forward. I think I — I think it's just because I'm lazy that I don't plan more, but I think sometimes when you just go for it, you find out quick, like what the best way might be. Like you can kind of brainstorm what might be possible earlier on and I think in some cases that's helpful, but it's just what I'm used to.

Ben credited a course in Discrete Mathematics as influential in improving his proving and problem solving skills, but stated that having a broad mathematical base is very useful.

I think just having a variety of knowledge in every area, or different areas of my life leads to tackle this sometimes or like the rate of change problems, there's different ways of doing them. I just think having so much math and having many different ways of doing different problems helps a lot.

This statement suggests that Ben was aware of the powerful role that conceptual knowledge plays in problem solving success. As has been reported earlier, Carlson and Bloom (2005) found that mathematicians’ problem solving success was also impacted by the depth and connectedness of their mathematical knowledge base.
The problem solving session

The problem solving session revealed that Ben was able to access key mathematical ideas. The first problem he attempted was the Square Paper Problem (Figure 78). While working through this problem, Ben demonstrated that he could orient himself to the problem and set up a solution strategy.

A square piece of paper $ABCD$ is white on the front side and black on the back side and has an area of 3 square inches. Corner $A$ is folded over to point $A'$ which lies on the diagonal $AC$ such that the total visible area is $1/2$ white and $1/2$ black. How far is $A'$ from the fold line?

Figure 78: The Square Paper Problem.

1. Ben: All right. So first can I draw it, $ABCD$.

2. Ben: And, they talk about the diagonal $AC$ — which lies on the diagonal $AC$, so we want to fold a corner of $A$ to a point, $A'$ which is on here somewhere. And when you fold it, it's going to look like this and they want total visible area, $\frac{1}{2}$ and $\frac{1}{2}$ — oh wow. That seems to be half and half, it would have to look like this, you know, so that would be, but that's not quite, so just somewhere around here.

3. Ben: So the area of here has to equal this area and we know that the total is 3 square inches so now that, and this is a square piece of paper so a square would be the sides are the same distance, 3, so that equals $\sqrt{3}$. All right. Now. To figure this out you'd have to turn from this point, how the area changes and let's see, $a$ here, $AD$ here and this. And they want to know the distance from the fold, this distance, so we'll call that $d$. They
want to know the distance that corresponds to area of black equals — yeah. All right. So this has to move down, so now I need to find the formulas for the area of this section, which is triangular.

Figure 79: Ben's beginning of Square Paper Problem.

[4] Ben: And this section and when this moves this way, this is going to decrease in some function so I am going to get this in terms of this moving and we'll know this also because we know this distance minus this distance and we can find, since we know this and this is also going to be square folding, then we'll be able to find this area which is this divided by 2. [laughs].

[5] Ben: So this is, $\sqrt{3}$ this is $\sqrt{3}$. All right, so we've got to get everything in terms of the $\sqrt{3}$ and $d$, preferably. So, area of, we'll call it $L$, the L-shaped region — is going to be $\sqrt{3}$, one is going to be 3 minus the area of the triangular region which we'll call $t$. And the area of this is going to be — so let's see, we need $d$. So I need to find this in terms of $d$. So if — find this divided by 2, in terms of $d$ and this is the $\sqrt{3}$ — (pause) — so, draw
just kind of a diagram, cause I need to find, I basically need to find this length and I'm given $d$ and I know that this is the $\sqrt{3}$.

*Long pause*

![Diagram](image)

*Figure 80: Balance of Ben's work on the Paper Problem.*

[6] Ben: In order to find this, this is the same as this which is the same as this and this is an isosceles, these are going to be the same, so it happens that there's a ratio – this is 45, so I know that the ratio is 1, 1, and the square root of 2?

[7] Ben: Yeah. So if you are given $d$, here, so this is, let's say $a$, $b$ and $d$, so $a^2 + b^2 = d^2$, and you know that in this case it's going to be $1+1 = \sqrt{2}d^2$, no, equals $d$, so is this right? $1^2 = 1^2 = 2$ squared and so the square root of 2 which is 2, right.
Ben: So, now how to get from the square root of 2 over to 1 and in order for that to happen I need to solve for one of the sides or either of the sides, oh this is okay, so it would be, all right, so side 1 is to what we want to know as — oh, okay, as the given $d$ is to $\sqrt{2}$. So $\sqrt{2}$ equals the given distance times what we want to know. One over, one over, this looks bigger. One over $x$, right, wait. This is what we'll do. We'll, since we have the similar triangles, we'll have the side, okay, $\frac{1}{\sqrt{2}}$ equals the side we want $x$ over, that would be $d$. $d$ is going to be given, so I've got the same thing, okay, we have $d = \sqrt{2} \times x$, right.

Ben: Okay. But we want to find $x$, so $x = \frac{d}{\sqrt{2}}$. And that's this side.

Ben: Does that seem right? $d$ —

Int: Is it?

Ben: $d$ is to $\sqrt{2}$, right, because then you would get 1 and that's the correspondence. Okay.

Int: Okay.

Ben: So, now I have that side, and I want to find the area of these little triangles, so it's $\frac{d}{\sqrt{2}}$, squared and then over to $\frac{d^2}{2}$ over 2 which is $\frac{d^2}{4}$.

So the area of this triangle is $\frac{d^2}{4}$ and $d$ is. $d$ is this distance, right. So $d$ is this distance.
In this passage, Ben engaged in sense making as he organized the relevant information and constructed a diagram [1-5]. He demonstrated the ability to map out a plan ([3 - 8, 14]) and accessed relevant mathematical knowledge [3, 6, 7]. He also monitored both his thinking [7, 10, 15] and his progress [9, 14]. However, as seen in the following passage, this monitoring failed to uncover a critical error in his thinking – he assigned the letter \(d\) to two different distances.

Ben: So now we know, so then, can we say that the other \(L\) is 3 minus the area of \(t\) and we decided that \(t\) is this

Ben: So area equals \(3 - \frac{d^2}{4}\) and \(t = \frac{d^2}{4}\) and they're on — those are equal but it doesn't seem like that's possible. So I either made an error or, oh wait, hold on, is that minus — oh, okay, I thought that was a plus.

\[
\frac{2d^2}{4} = 3 \cdot 2d^2 = 12\ so\ d^2 = 6.\ \text{So it's} \ \sqrt{6}.\ \text{No, the} \ \sqrt{6}.\ \text{If this is} \ \sqrt{3} \ \text{and this is} \ \sqrt{3},\ \text{then,} \ 3+3, \ \sqrt{3} \ \text{plus, that's weird.}
\]

Ben: Well — like if I were to find this diagonal?

Ben: Like the one for that diagonal...it would be \(\sqrt{6}\), right?

\[
\left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 = 3 + 3 = 6\ \text{then you take the square root of that.}
\]

Ben: So at that point, this distance would be here so it would fold in the middle but if that were true, like you wouldn’t, like technically, that's...doesn't work out though.

Int: Okay.
Ben: I'm not sure if you flipped it over. Nope, cause then there'd be a big blank. So that would be —

Int: So what do you do when you're stuck in a problem?

Ben: I'm probably not really stuck right now because it made sense to me in my head, so I'd probably go back to it and check for errors and then if I did it again, and it didn't come out again, I'd probably look at it and see if there's any kind of, I don't know, weird relation that can't make this happen. Like you should be able to, well, I don't see why you shouldn't be able to fold, okay. Well it doesn't really matter, I can't go past here so, let's see. [long pause as he stares at the work in front of him] Like I don't, wait a minute. Because we're looking at this. [another long pause] Well, perhaps maybe, I'm not sure, but perhaps maybe this. But, if it is going to continue like that you have to look at it and see, if you keep doing something that can't happen, then maybe it can't happen. Like I'm not exactly sure but perhaps this can never equal this, just by the nature of the problem.

Ben’s conceptual knowledge base afforded him the tools needed to construct and execute his solution [16-19]. This episode also revealed that he could be persistent in his efforts to solve problems. When the interviewer gave him an opportunity to stop in [23], Ben said he really was not stuck [24], and continued to think about the problem. He also expressed confidence that he could solve the problem [24] because "it made sense to me in my head." Ben reflected upon his progress as he worked through the rest of the problem, realizing [in 17] that there was an error although he was unable to find it. This behavior
of reflecting upon the products and processes of his mathematical work was unusual so
early in the semester in comparison to his peers.

    The next excerpt shows his work on the Least Common Multiples Problem
(Figure 81). As with the Square Paper Problem, Ben was able to access relevant
mathematics, such as knowledge about factors and multiples as he oriented himself to the
problem.

List all the integers less than 100 that are divisible by 2, 3, 4, 5, and 6

Figure 81: The Least Common Multiples Problem.

[1] Ben: Well if the number is divisible by 6 it's also divisible by 2 and 3.

[2] Ben: And if it is divisible by 4, it's also divisible by 2, so — I hope this is
good — I'm just going to disregard those numbers.

[3] Ben: And I guess it's really these three that have a problem, because if it is
divisible by 6 it is not necessarily divisible by these two, but just because
it's — okay, that's good. Because if it is divisible by 5, it doesn't mean it's
divisible by 2 necessarily but it also has to be divisible by 6, which is
divisible by 2, so I think it's pretty safe to throw this out.

[4] Ben: And this out. So, well. So — 4 times 5 is 20 times 6 is 120. 120 is
greater than 100, and so that doesn't count, but the greatest common – the,
the least common multiple?
Ben: The least common multiple of those three numbers would seem to be, that, except for, 4 times 5 is 20, times 3 is 60. And 60 is divisible by all of those. Let's see, what else. Since we know 60, we can't go up by 60 which would be nice. 30 is not divisible by 4. So here we have, we have a 30 here, we have a 24 and — which one did I have down — here we have a 20. So we can't go up by 20s, because it's not divisible by 6, and we can't go up by 24s, because it's not divisible by 5. We can only really go up and down by 30s cause it's the only thing all, 4. Oh, that's not divisible by 4, right. Yeah, that's not divisible by 4. So we can't move in any of these directions. So, it seems, I seem like I'm stuck already. Is there anything else? Like we said earlier, 2 and 3 go into all of these, 2, 3, well, except for this one. But that one's included in the 6, so if we rule out 6 we don't do that — it seems like the smallest way that we can do even 2 of these, we can't increment from here. The smallest one we can increment is the one that already 3 is divided into it which is 60.
In this passage, Ben immediately accessed his conceptual knowledge of divisibility and least common multiples and proceeded to use that knowledge to generate and test conjectures. For example, he made a conjecture about disregarding the integers 2 and 3 [1], and then he stopped [3] and checked the reasonableness of that conjecture before moving on. He checked the validity of his solution and appeared to engage in monitoring. At the same time, he was not as confident as one might expect. In passage [5] he claimed that he was stuck, and in [7] he did not express certainty that he had a solution.

The third problem Ben worked on was the Mirror Numbers Problem (Figure 83). In this passage, Ben became confused as to the meaning of the variables he chose.

Two numbers are "mirrors" if one can be obtained by reversing the order of the digits (i.e., 123 & 321 are mirrors). Can you find:

(i) two mirrors whose product is 92565

(ii) two mirrors whose sum is 8768

*Figure 83:* The Mirror Numbers Problem.

[1] Ben: Okay. Well so, they're not consecutive then you can't exactly write them as, like, $a \cdot (a + 1) \cdot (a + 2)$. Well that's three separate variables. It's kind of like $abc$ and $cba$.

[2] Ben: And those are in basically digits so $abc$ times $cba$ — this is helpful, with two $b$'s lining up.
Ben: Has to equal this. So... $a^2 \cdot b^2 \cdot c^2$ — it's product right? Times $a^2 \cdot b^2 \cdot c^2$ equals 92565. So then you're coming up with something like that, I guess.

Ben: And we know that you have to find like factors of this, for this to work, and they have to be squares, which is fantastic. So factors that square out of this. We'll just factor and try to find them, I guess. [laughs]. These are weird problems.

Int: [laughs]. Carefully selected.

Ben: Okay, let's start with 5 — let's use the obvious one.

Int: You can use the calculator if you want.

Ben: All right. 92565, 5, so we have the 5 and so this is 5, and I want to check if it's going to be divisible by 3, which it is. We have a 3.

Int: Okay.

Ben: All right. And that is also divisible by 3 — so we’ve got another 3, which equals this.

Int: Okay.
Ben: And that is not divisible by 3. I wonder if it's going to be divisible by some big old prime. So we have like 1 of them, possibly. It's not divisible by 5, then we don't have a second square for this one. Like, there's no way you could get a 5 out that.

Figure 84: Ben's work on the Mirror Numbers Problem.

Ben: So you can't have a second square — and regardless, if you don't — if you even, you could find another square. If you can't find something to square this, then you might be stuck. Now I'm wondering if it's prime
factorization that will help you with this because you have to pull out some squared numbers and you have to pull out three of them, and if you could come up with a list of, like, primes, you could turn, you could use those two to find some squares but. [pause] I don't know what else could go into this. Really like nothing [laughs] — nothing that's smaller than, smaller than 10, I guess.

In this passage, Ben was observed engaging in sense making [1-2] but in [3] he decided to treat what he had identified as “digits” as if they were factors. His subsequent plan [4] was based on that error, thus his efforts to execute that plan [6-13] were ineffective. Although statements such as, “I’m wondering if it's prime factorization that will help you” [13] could be evidence of self-monitoring, such monitoring did not uncover his mistake regarding the representations he chose. In this case, he gave up when offered the opportunity [15], and elected to try the second part of the problem.

[14] Int: What about (b), want to try that one?

[15] Ben: Sure. Okay, so same kind of thing, the sum, so, it would be

\[ abc + cba = 8768. \]

So, I mean, you could say that \( a(bc + cb) = 8768 \). \( a \) times what? Oh, well, I guess, that's an \( a \), right? No, what'd I do, these are numbers. [Laughs]. That's kind of freaky. So but all when you are doing is addition, it's kind of like, it's more like \( a + b + c \) which is like — so it's \( a \times 100 + b \times 10 + c \), because these are like — this is one number, it isn't like a number times a number.
[16] Ben: Plus 100c + 10b + a so we have 100 times b + c plus 10 times b + b and then plus c + a = 8768.

He continued to work on this path for several minutes.

[17] Ben: So if I subtracted 8 of those, oh, so 8 = c = a, like whatever, and then 8768 - 8, I believe, it's going to be 067 what's that, 9? Say I got 960, so 20 would be equal to 7960 [entered numbers into calculator] fool, cause it, well not doing it right though because it would have to be a bigger integer, so I'd have to subtract, I'd have to subtract more, but I don't know how big a and c are going to get, cause a and c are only one digit.

[18] Ben: Oh, I see. [laughs]. So there's no way. That's stupid. That sucks. Cause there's no way you could have it a 999 and then if you added twice it doesn’t correct for that so we can't do (b).


[20] Ben: Hold on. So 999. Oh, I think that could happen. Yeah. 98, 998, wait a minute, the biggest number is not over that so this is still possible. 998 [used calculator,] yeah. I don't know if there's an answer though.

In the second part of the Mirror Numbers Problem, Ben recognized his error in thinking about what his variables represented [15] and reconstructed the problem so that the variables represented digits. Even though this indicated some level of monitoring, he did not connect this error to the difficulties he had experienced in the first part of the problem. He set up a plan [16] and attempted to carry it out. Although he became frustrated [17] with the problem, his thinking led him to realize there could be no solution
for (b). While he stated in [18] that there was no solution, his confidence appeared to waver as he indicated he was unsure whether a solution existed [20].

In summary, Ben's problem solving session revealed that Ben was able to access key mathematical ideas. Unlike Amy, he did not rely on "guess and check" as a heuristic, yet he displayed few heuristics other than creating a diagram in the Square Paper Problem. There was evidence of monitoring as he worked, but that monitoring was not always effective in revealing errors in his work. His work on the Square Paper Problem and the Mirror Numbers Problem revealed that he seemed confused by the meaning of the variables he selected, and this confusion appeared to interfere with his ability to find solutions for those problems. Earlier in his interview, Ben revealed that he found the various notations used in calculus confusing, suggesting that notational choices could pose difficulties for him. In general, he appeared confident in his approaches to the problems presented, supporting his earlier claim that he was confident in his abilities. He also claimed that he was persistent in his problem solving efforts. In the problem solving session he was reluctant to let go of the Square Paper Problem, however he did give up on first part of the Mirror Numbers Problem.

These data together with the pretest and VAMS results suggested that Ben was able to access his mathematical knowledge when confronting novel problems. He appeared to possess the conceptual and procedural knowledge needed to solve these problems. He also appeared to be confident in his mathematical abilities. His views about the practice of mathematics and problem solving were also aligned with those of the mathematical community. However, these data suggest that Ben experienced some difficulty interpreting the notations and variables he chose. Despite these difficulties, his
performance on all three of the measures (pretest, VAMS, interview) revealed that he was one of the stronger students in the class.

Following Ben Through the Semester

This section provides a description of Ben's progress and participation over the semester, including his individual work as well as explanations and interactions with his group members, classmates, and instructor. The research data has been organized into three sections, the first 5 sessions, the middle 10 sessions, and the final 8 sessions, to allow for investigation over shorter periods of time as well as across the entire semester.

Early in the Semester: Sessions 1-5

During this first phase of the semester, Ben's mathematical behavior was similar to that revealed in the pre-instruction measures.

Problem sets

Ben's work on the problem sets was generally good. He solved the majority of the assigned problems correctly and was able to articulate his thinking in writing. He was also able to identify the key concepts used to solve the problems. Unlike Amy, Ben demonstrated persistence by submitting complete problem sets.

Analysis of his work on the problems revealed that he had a tendency to look for quick and simple solution approaches. The episode below is taken from a discussion of Ben's solution to the Toothpick Problem (Figure 85). With some coaching, Ben was able to share his thinking about the problem. However, the problem statement asked for the general term of the sequence, but Ben's solution (Figure 86) stopped when he discovered a recursive formula for the next term.
6.) For the following sequence made of toothpicks, let \( S(n) \) represent the total number of toothpicks in the \( n \)th figure. Find the general term for the sequence.

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\[ \text{Figure 85: The Toothpick Problem, Problem Set 1, #6.} \]

[1] Inst: Alright. So now let’s look at number six. We’ve got two answers. So Ben wrote his recursively. Do you think that’s fair?


[3] Inst: As an answer to the question?


[5] Inst: OK. Ben, tell us what your thinking was.

[6] Ben: Oh. Umm, well I just got lazy and just pictured, you know, the next step and saw what you had to add on to it and that’s what I came up with.

[7] Inst: So how’d you come up with \( 4n \)?

[8] Ben: If you count, there’s like four sections of \( n \) number of toothpicks, like if you were to look on \( S(3) \), when you look above it, there’s the 3 vertical ones — I don’t know — right above the square, and the three horizontal ones and the 3 that cap it off on the top and then the 3 that cap it off on the side. So it’s 4 times 3 over \( 4n \).
Inst: OK.

Ben: And that adds, and it’s the same as 4, 2, 4 times 1.

Inst: So what’s the weakness of a recursive formula?

Ben: You’d have to calculate each one to get it all the way up there.

Inst: It’s pretty much a drag. But will his formula still give you the right answer?

Class: Yes

At first Ben thought his written work (Figure 86) was self explanatory [6]. When the instructor prompted him with a specific question [7] he was able to clarify his reasoning for the class [8-10]. His explanation reveals that he was able to reason about the patterns created by adding toothpicks to the model, and then use that reasoning to construct a formula for generating successive models. The instructor then pointed out the weakness of his recursive formulation while at the same time asking the class to confirm that Ben’s formula would generate the same sequence.
In other instances, it appeared that Ben committed himself to a plan of action without considering other approaches or reflecting on the viability of the first idea that entered his mind. The following two examples illustrate this tendency. In the Employee Rate Problem Ben immediately mentioned that he could solve the problem by computing the average rate of change and he proceeded with this solution approach without considering whether this approach was optimal for answering the question. In the following two examples, he seemed to pursue the first thought that came to his mind. The first example is the Employee Rate Problem (Figure 87).

2.) A company estimates that in $t$ years the number of its employees will be $N(t)$, where $N(t) = 1000(0.8)^t$

   a.) How many employees does the company expect to have in 4 years?

   b.) At what rate is the number of employees expected to be changing at 4 years?

*Figure 87: The Employee Rate Problem from Problem Set 2, #2.*

Although he effectively accessed his knowledge of average rate of change to compute a rough estimate for the rate at four years (Figure 88), the problem called for an instantaneous rate of change. His statement, "Calculus could be used but that's just too easy," suggests that he realized that calculus might be a more appropriate tool, yet he continued with his original approach without reflecting on how calculus might be used to solve the problem.
In the following problem (Figure 89), Ben again pursued an algebraic approach without considering concepts or tools from calculus. When solving this problem he constructed a correct algebraic representation for the cost of the fence. However, he followed this by following an approach that was guided by his stated knowledge that the cost equation is maximized when the shape is a square (Figure 90).

2. A certain farm allots $1000 for fencing a rectangular area that is to abut a highway. Because the fencing on the highway side must be attractive, it costs $4 per lineal foot. The other three sides of the area are fenced at $2 per foot. What are the dimensions of the rectangle that maximizes its area?

Figure 89: The Highway Fence Problem, Problem Set 4, #2.

He initially constructed an equation for the cost of the fence. He then used this equation along with a guess and check approach to find values for the length and width by simultaneously subtracting a number from one dimension and adding it to the other. Even though his final answer was close to the result he would have found using calculus, it was not exact, nor was it an efficient approach to solving the problem. He recognized the difficulty in this approach by saying that, "This problem is frustrating in that it is hard
simultaneously keeping both equations in mind.” It is not clear why he didn’t take the time to consider using tools of calculus.

---

**Figure 90:** Ben’s work on Problem Set 4, #2.

In other instances, Ben’s strong conceptual knowledge and mathematical intuitions allowed him to construct efficient solutions. When he solved the Composition problem (Figure 91), for example, he produced an efficient solution and provided justifications that revealed his effective use of conceptual knowledge (Figure 92).

1.) Consider the function \( f(x) = \frac{cx}{2x+3} \) where \( x \neq -\frac{3}{2} \). Find all values of \( c \) (if any) for which \( f(f(x)) = x \)

**Figure 91:** The Composition Problem from Problem Set 3, #1.
His mathematical understanding of function composition and rational functions allowed him to recognize that for the composed function to produce $x$, the denominator must be a constant and it must be equal to the constant in the numerator. This insight allowed him to bypass some tedious algebraic computations. It appeared that in this case (unlike the previous examples), he was able to reassess the problem situation after he simplified the composed function.

![Figure 92: Ben's work on the Composition Problem from Problem Set 3, #1.](image)

In summary, the written work submitted during this instructional period suggests that while Ben was generally successful in producing reasonable solutions to these tasks, he displayed a tendency to execute solution strategies without carefully considering alternatives. This behavior, as illustrated in his solution to the Highway Fence Problem,
was less efficient and accurate. However, at other times, when working problems that he
possessed strong conceptual knowledge, he was able to efficiently identify useful
conceptual resources for completing the problems. He primarily employed diagram
sketching as a heuristic to help him visualize the problems (Figure 86 and Figure 89). He
appeared to persist in completing complex problems, as evidence by the fact that his
written assignments also contained complete solutions. As with the mathematicians in the
Carlson and Bloom (2005) study, Ben expressed both frustration and joy when
completing his assignments (see handwritten comments on the Highway Fence Problem,
Figure 90). While there was evidence that he monitored his processes and products, the
effectiveness of his monitoring was mixed. For instance, in the Employee Rate Problem,
his reflection did not produce a change in his solution path, whereas in the Composition
Problem it appeared that his reflective behaviors led to an interesting and elegant
solution.

Class work

Ben was an active participant in small group discussions. On the first day of class,
students solved the Magic Square Problem (Figure 93) in their groups and presented their
solutions to the class.

In your group, create a 3x3 “Magic Square” that uses the numbers 1, 2, 3, 4, 5, 6,
7, 8, & 9.

Figure 93: The Magic Square Problem.

When presenting his group’s solution, the instructor asked him how he determined
the number placed in the center of the grid. Ben responded as follows:
And then it just seems kind of logical that if every other number has to add up with that middle number, well most of them do, then you’re going to have one big and one small one and one in the middle. So the only way you could make sure that they all kind of originated around that center number there you’re going to have to relate to the number in the middle.

In the passage above, Ben referred to the pattern he noticed in the sequence of numbers in the problem. His search for patterns and his ability to make and test a conjecture about the number in the middle square contributed to the groups’ construction of a correct solution to the problem.

When working on the Fly-on-the-Wall Problem (Figure 94), Ben was able to quickly form an accurate image of the problem situation, as revealed in his discussion with Callie.

Two walls and the ceiling of a room meet at right angles at the point \( P \). A fly is in the air one foot from one wall, eight feet from the other wall, and nine feet from the point \( P \). How many feet is the fly from the ceiling?

*Figure 94:* The Fly-on-the-Wall Problem.

[1] Callie: How do I find this from that right there?

[2] Ben: So if you’re one foot from this wall, your one foot this way. And if you’re 8 feet from this wall, your 8 feet this way. So you’re at a point right here. So, you need to find the height so that this from this point is 9. So here’s 1. Here’s 8.

[3] Callie: Where’s your ceiling?

[4] Ben: This is the ceiling.
Karin: Up. So it’s like that distance from the ceiling.

Callie: Oh, okay…

Ben: Well it’s straight up. It’s this distance right here. This is 9 from the point. This is one foot away from one wall. And this is 8 from the other.

Callie: Ok, so how do we find that?

Ben: Here’s the triangle with the right angle. So you need to find this to find that. To find this, but you have a 1 here and 8 here. This is also a right triangle.

In this episode, Ben helped Callie visualize the place where the fly was located in three-dimensional space [2, 4, 7] and he suggested a strategy for finding the required distance [9]. His contributions to the group suggests that he was able to: i) effectively orient himself to the problem by visualizing the problem, ii) access appropriate conceptual knowledge to determine the relationships between the quantities in the situation; and iii) appropriately use that knowledge to consider a viable solution approach.

What is the highest score that you cannot get on the dartboard shown below if you may throw as many darts as you wish? Explain

![Figure 95: The Dartboard Problem.](image)

Other times, his contributions did little to help his group progress toward a solution. In the second class session, students were presented with the Dartboard Problem
When introduced to the problem, Ben initially related it to one he had seen before.

[1] Ben: See, I had a problem like this and it drove me absolutely crazy!


[5] Ben: This is going to drive us all nuts

[6] Callie: If you can throw as many darts as you wish?

[7] Ben: So, like, there’s going to be a number that you can’t get – is what they say.

In this passage, Ben admitted to his group that he had worked a similar problem [1], which can be a useful heuristic. However, his contribution related only how hard he found the problem, not ideas or strategies for finding the solution. The group was making ineffective attempts to work on the problem until the instructor intervened. While targeted questioning by the instructor helped move some group members toward a solution, Ben stayed stuck in his previous experience with a problem of this type.

[8] Ben: See I had a problem — remember I was talking to Dr. Smith about that problem?


[10] Callie: What did Dr. Smith say?


[12] Callie: Are you serious?
Ben: Yeah.

Callie: Dr. Smith is a genius.

Ben: I’m sure he’s done other problems like it before. He just couldn’t think of how to do it.

In the excerpt above, Ben’s perception of the problem’s difficulty appeared to contribute to his unwillingness to attempt the problem. This suggests that his affective reaction influenced his approach (unwillingness to engage in sense making) when confronting this task. Although the group eventually solved the problem, Ben's contributions to the discourse did little to advance the solution and may have in fact hindered the group's efforts for a time.

In summary, Ben’s contributions during the group problem solving sessions were generally productive, although his affective response to the Dartboard Problem appeared to present an obstacle for him in making sense of the problem. As with his individual tasks reported earlier, he typically read and made sense of the problem; then attempted to access resources that he thought would be helpful for the solution approach. In the Magic Square Problem, he searched for patterns and was able to check his conjectures. He also exhibited that he was able to accurately visualize the problem context, as was the exhibited in his approach to the Fly-on-the-Wall Problem.

Extended Analysis Tasks

The first Extended Analysis task assigned to the class involved the Catching up Problem (Figure 96).

After finding a numerical solution for the problem, the students were expected to parameterize the problem, explore the structure, and create an isomorphic problem – a
problem that had the same mathematical structure as the Catching up Problem. The numerical solution for the problem, as well as some of the initial analysis, was conducted in class with the support of the instructor.

Person A sets out in a car going at 50 mph. Starting 3 hours later, person B tries to catch up. If person B goes at 75 mph, how long does it take to catch up?

Figure 96: The Catching up Problem.

As part of the in-class exploration, students were asked to replace the catching up time and catching up speed with parameters and construct a new function representing the catching up time as a function of the catching up speed. Students were then instructed to graph this new function. In the following passage, Ben and Callie were making sense of the graph they plotted.

[1] Ben: So if all tangents are going to be negative that means it’s always decreasing. The distance, well see, here I thought the rate of the distance was changing. The distance was – shorter? Oh it’s always decreasing it’s just a smaller decreasing.


[3] Ben: Like here it would be, whatever, negative a million.

In [1] and [3], Ben appeared to be using his conceptual knowledge to make sense of the new functions by attending to the varying quantities represented in the graph. Later, the instructor asked them to find the rate of change of that function when the catching up speed is 75 mph. In the next episode, Ben was able to evaluate the derivative at 75.

[1] Ben: It’s negative point 2 4 miles per hour.
Callie: That’s how much they’re gaining on each other at that point in time.

Inst: OK. Can you put it in terms of, since the speed of car B is the independent variable, can you put it in terms of, “If I do this to the independent variable this is how the dependent variable reacts?” Can you get it in that kind of thinking?

Ben: If the speed is at 75 —

Callie: The speed of car B is independent?

Ben: Well it’s the time that’s dependent then.

Inst: Right. We’re catching up time.

Ben: So the time is decreasing by point 24.

Inst: Point 24 what?

Ben: Hours.

Inst: OK.

Ben: Per miles per hour.

Ben was able to arrive at a numerical answer [1], but he required some coaching from the instructor [3, 7, 9] to place that answer in the context of both the function in question and the original problem.
The take home portion of the extended analysis asked the students to find the catching up time \((t)\) as a function of the difference in the velocities.

\[
\text{catch up when } \quad 3V_i + V_i t = tV_2
\]

So \(t = \frac{3V_i}{V_2 - V_i}\) could be substituted for some variable \(X\) (time ahead) but \(3V_i\) is equal to the distance that needs to be made up.

So if interpreted differently the equation looks like

\[
t = \frac{\text{distance to make up}}{\text{difference in velocity}}
\]

This makes sense because any ground covered by both cars at \(V = V_i\) makes no difference because it is \(X\) extra velocity \((V_2 - V_i)\) that closes the gap.

It's like if you could film an overhead shot with \(X\) camera following \(V_i\) it would look like \(V_2\) was moving towards \(V_i\) at exactly \((V_2 - V_i)\) and it would take exactly \(\frac{\text{distance to make up}}{\text{difference in vel}}\) time.

Graph looks like this

\[\text{Figure 97: Ben's work on the extended analysis of the Catching up Problem.}\]

Ben correctly constructed this function (Figure 97), and his explanations suggest that he understood the function and could interpret it in the context of the problem setting. However, his sketch of the graph was very general, showing no scale or inscriptions other than the axes and a curve.

The isomorphic problem Ben submitted did not share the same mathematical structure as the Catching up Problem. He wrote:
Another problem that might have the same mathematics involved would be asking if you could split the distance between you and a friend in half would you theoretically ever touch?

Analysis of his justification that the two problems were isomorphic (Figure 98) revealed that Ben was attending to the general shape of the graph rather than the mathematical structure of the problem and the relationships between quantities.

**Figure 98:** Ben's justification of his isomorphic problem.

His work on the extension assignment suggested that Ben was able to access relevant mathematical knowledge to create and make sense of new functions generated from the Catching up Problem. His isomorphic problem and subsequent justification, however, suggested that he attended only to the superficial similarities in the two functions.

**Summary of sessions 1-5**

Ben's work in this early phase of the semester corroborated findings from pre-instruction measures; he was able to access key mathematical concepts when solving problems, and appeared to be fairly confident and persistent in his efforts. However,
analysis of data from the first five sessions revealed certain aspects of his mathematical behaviors that had not been observed previously. Ben's work, both individually and while interacting with peers, was uneven. Sometimes he committed himself to a plan of action with little reflection on his products or processes; other times, he submitted elegant solutions that relied on a strong understanding and ability to use central ideas of algebra. He was frequently observed employing effective problem-solving behaviors within his group; however, there were instances when he elected not to engage in making sense of the problem.

**Transitions: Sessions 6-15**

In class sessions 6 through 15, changes in Ben’s mathematical behaviors became evident, particularly in the areas of planning and monitoring. Changes in his work with the assigned problem sets, his interactions during class, and his work on extended analysis tasks are discussed in the following sections.

**Problem sets**

In the Balance Scale Problem (Figure 99), Ben clearly identified and analyzed the different cases needed to find a solution to part (a), but his work on part (b) fell short of a proof. In the excerpt below, he explained his reasoning about the generalization to the class.
3. Solve the following problems assuming you only have a balance scale at your disposal.

(a) Suppose you have 9 coins that are equal in appearance. Eight of these coins are identical in weight, and one is lighter. Explain how you can use the balance scale to identify the lighter coin by using the scale twice.

(b) Generalize this result. That is, if you have $3^n$ coins and all but one of them weighs the same; prove that you can find the odd coin by using the scale $n$ times.

Figure 99: The Balance Scale Problem from Problem Set 5, #3.

[1] Ben: If you write out 3 to the — whatever, you can always pull out the 3. Then you can always have 3 groups and then every time you do it, you divide the groups by that factor of 3. So you go down by the exponent every time, so you’re going to do it exactly $n$ times – dividing by 3 – because there’s always that 3 you can bring out.

[2] Inst: Is that as convincing an argument as an induction proof? If you were the teacher, would you accept it?

[3] Callie: Yeah, if you didn’t say “use induction.”

[4] Inst: Why didn’t you use induction?

[5] Ben: Well, isn’t it — oh, so you would start, so that would be from the bottom up, if you were to use induction?


His explanation draws on his conceptual knowledge about exponents [1], but it lacked the rigor of a proof by induction [2-7].
For the most part, Ben’s work on the assigned problems was well thought out and his tendency to seek simple solutions (seen in the first segment of the semester) was not as evident.

2. Consider the family of quadratic functions given by \( f(x) = (m^2 - m + 1)x^2 - (2m)x + 1 \)
where \( m \) is any real number. For what values of \( m \) will \( f(x) \) always be positive?

Figure 100: The Quadratic Family Problem from Problem Set 6, #2.

For instance, in his work on Quadratic Family Problem (Figure 100), Ben (unlike Amy) carefully analyzed the quadratic function and used that knowledge to guide his solution. In his solution (Figure 101), Ben employed the setting sub-goals heuristic by addressing the two necessary conditions that must hold for the function to be positive: the coefficient of \( x^2 \) must be positive and the \( x \)-coordinate of the vertex must also be positive.

First off we want this quadratic equation to open up, because if it opens down then undoubtedly it eventually will end up regale.

\[ (m^2 - m + 1) > 0 \]

Well this quadratic equation opens up and has its vertex above the \( x \)-axis so it is always positive. Therefore we don’t ever have to worry about it equaling zero down because \( m^2 - m + 1 \) is always positive.

Next all we have to do is make sure the vertex is above the \( x \)-axis.

So we know the \( x \)-coordinate of the vertex is \( \frac{-b}{2a} \) or \( \frac{-2m}{2(m^2 - m + 1)} \) and we want \( f \left( \frac{-b}{2a} \right) \) to be positive. Ugh...

Figure 101: Ben’s solution to the Quadratic Family Problem.
In the Remainder Problem (Figure 102), Amy and Ben both chose to use modular arithmetic to answer the question (Figure 103); however, Ben’s solution was correct. It is interesting to note that while Ben was able to make use of his knowledge of Number Theory to make sense of the problem and construct a viable solution path, it appeared that he needed to use numerical examples to convince himself that he had arrived at a correct solution.

4. When an integer is divided by 15, the remainder is 7. Find the sum of the remainders when the same integer is divided by 3 and by 5.

*Figure 102*: The Remainder Problem from Problem Set 8, #4.

*Figure 103*: Ben’s solution to the Remainder Problem from Problem Set 8.

During this segment of the semester, Ben’s written explanations were clear and understandable and supported his computations. His conceptual knowledge of the mathematics required afforded him the ability to make sense of problems and construct viable solutions as evidenced by the fact that many of the problem sets Ben turned in during the transition period were free of errors. He employed a greater variety of heuristics than were seen in the first phase of the semester, including examining cases in the Balance Scale Problem and setting sub-goals in the Quadratic Family Problem.
Class work

Ben’s use of self monitoring and evidence of mathematical integrity became apparent during group problem solving sessions.

When Ben’s group started to work on the Triangle Problem (Figure 104), Dora stated that they needed to look for the centroid. The whole group followed that path for quite some time. In the excerpt below, various members of the group were trying to construct the centroid of a triangle.

You are given a fixed triangle $T$ with base $B$ as below. Show that it is possible to construct, with a straightedge and compass, a straight line that is parallel to $B$ and that divides $T$ into two parts of equal area.

![Figure 104: The Triangle Problem.](image)

[1] Ben: We have to show that the centroid is the point that this line passes through to create equal the area. Unless we already know that. But that’s just a guess.

[2] Dora But it has to be.

[3] Ben: Because it has an angle bisector. It’s where the angle bisectors meet.


[5] Ben: But you don’t have to know that the centroid’s the point where this angle bisector intersects it. It divides it. So that $A$ plus $B$ equals $C$ plus $D$. 
So this area equals this area. Unless that’s something that we know already. That was just a guess. I’m just throwing that out there. I don’t know if that’s true or not. I don’t want to be getting us off on some kind of crazy tangent.

[6] Ben: Well then you have to find two points to make them parallel. You know what I’m saying?


[8] Ben: When we find this point we’ll have to find it perpendicular to here to show that it’s parallel [in audible].

[9] Shania: But we can.

[10] Ben: But see now we have to know that that point is on the line that divides that area. Do you think that’s true? Or is it all just a guess?


[12] Ben: Because we can sit there and prove it. So how would you prove it?

[13] Shania: You’re saying that the centroid is the center of, wait; the centroid is just intersection of all your angle bisectors?

[14] Ben: And we think it’s the point that resides on this line. That’s the hypothesis.

[15] Ben: It’s just a complete guess. The only reason I think it’s that is because that’s the way that would be the easiest way to solve this problem.

Although Ben agreed with Dora’s conjecture, in lines [1] and [5], he pointed out that they would need to prove that the centroid is on the line that divides the triangle in half. He reiterated that they were proceeding with an assumption rather than a known fact in [10]-
This suggests several things. First, it appears that Ben was attempting to engage in a conjecture-imagine-evaluate cycle, which had not been witnessed before this. Additionally, verbalizing the necessity of proving their conjecture as well as his concern that they may be heading off on a “crazy tangent” [5] indicates that Ben was actively evaluating the effectiveness of the solution path discussed. Finally, his concern that the conjecture be proved before moving on suggests mathematical integrity. In [6] and [8], he also revealed that he was planning the steps they would need to in order to make the construction.

As the conversation in the group progressed, some members suggested that they might need one of the other centers of a triangle, but Ben continued to voice his concerns about the validity of their argument. Later in the class period he said:

If we’ve got a formula for area we can determine what point we need to construct. Because, I don’t know if these points are even valid. You know? It’s an experiment to see if we can figure this one out.

Work on the Triangle Problem continued into the next class session, as did the group’s continued focus on the centroid of the triangle a key to the finding the solution to the problem. Ben continued to voice his concern about how they could prove that the centroid would be on the dividing line. In an effort to focus their thinking, the instructor encouraged the students to consider the fact that a line parallel to the base creates two similar triangles. She then suggested that they first validate their conjecture empirically. When Ben constructed the centroid of the triangle (Figure 105) and then a line parallel to the base and measured the areas of the two regions, he found that the areas were not equal. He told Callie, “This point might not be the point.”
In the following excerpt, the instructor urged Ben to explain his focus on the centroid.

[1] Inst: So, Ben, what is your thinking?

[2] Ben: I don’t know. For construction purposes, it would be nice if it was one of the crazy points of a triangle – you know – if it would lie on that line.

[3] Inst: I’m just wondering where the “big hint” would fit in.

[4] Ben: Exactly. And through analyzing, it doesn’t look like this one works, so now we’re thinking –.

[5] Inst: Because I don’t see any similar triangles over there. So, how did your calculations work out?

[6] Ben: This was estimation, but it didn’t work out too well. But then, if you think about similarity, you still have to be able to draw a line.


[8] Ben: So, you need some kind of measurement. So that’s why it would be easier to say “well, if you could find this point” I’m
trying to think of construction first – see how it would be easier to
do that — and then try to go with a plan. Because I can’t exactly
measure this certain distance.

[9] Inst: Sure you can. You just can’t use a ruler to measure a certain
distance. So you are trying to find where the line should go first,
right?

In the preceding interchange, the Ben not only admitted that his conjecture did not appear
to be valid [4 and 6], but he also offered some insight into why that conjecture was so
attractive to him [2-8]. It appeared that “trying to think of the construction first” [8]
prevented him from considering alternate solution paths. The instructor was able to
refocus his thinking on making use of the properties of similar triangles. After this
interchange, the group set aside the centroid conjecture and began setting up the
proportional relationships needed to solve the problem.

When asked to reflect on his experience with the Triangle Problem, Ben showed
insight into his problem solving behaviors when he wrote the following:

I failed to use other strategies and spent a lot of time on a hunch that led
me nowhere. My strategy was effective in finding the easiest solution that
required the least amount of work. The solution held no mathematical
water but had it worked, it would have worked fast.

In the written reflection, he also admitted that he tended to “get hunches quickly” and
often did not “spend time thinking about the givens of a problem.” This was behavior that
had been observed in Ben, especially in the first segment of the course. In this case, even
though his “hunch” kept him stuck for a while, he also spent a great deal of time thinking
about whether his conjectures “held mathematical water,” and (with support from the instructor), he was eventually able to set aside his hunch.

Ben continued to voice his consideration of the validity of his conjectures. When working on the Inscribed Triangle Problem (Figure 106), Callie and Amy decided that the figure should be an equilateral triangle, Ben reminded them that they would need to prove that conjecture.

Three points are chosen on the circumference of a circle of radius \( r \), and the triangle containing them is drawn. What choice of points results in the triangle with the largest possible area? Justify your answer.

*Figure 106: The Inscribed Triangle Problem.*

Group problem solving sessions during this phase of the semester revealed that Ben’s problem solving behaviors were evolving. Ben began to not only make explicit his conjectures but to consider how to ascertain the validity of such conjectures. These episodes also show Ben critically evaluating suggested solution paths. Written reflections revealed that Ben gained insight into his tendency to seek easy solution paths and the general weakness of that strategy. These data reveal expressions of on-line monitoring and mathematical integrity, which had not been observed prior to this time.

*Extended Analysis Tasks*

During class sessions 6 through 15, students completed and presented two Extended Analysis Tasks. They first extended the Evaporation Problem (Figure 107). This classic problem yields the surprising answer that 50.05% of the water has evaporated. Unlike Amy, Ben correctly assessed the relationships indicated by the
problem statement. In the excerpt below, Ben’s thinking is revealed as Callie and Amy repeatedly questioned him about his reasoning and his chosen representations.

<table>
<thead>
<tr>
<th>A substance is 99% water. Some water evaporates, leaving a substance that is 98% water. How much of the water has evaporated?</th>
</tr>
</thead>
</table>

*Figure 107: The Evaporation Problem.*

1. Callie: I got that and that [pointing], so what did you do [here]?
2. Ben: So then you subtract some of the water. So the 98% of this is water and 2% of it is still — but see how the amount of this stuff doesn’t change? The amount of water changes because it’s the water that’s evaporating.
3. Callie: Right. But how come — are you sure those x’s are the same?
4. Ben: These xs are the same. These w’s are not. Is that right?
5. Amy: What does this statement say? 98% of the total minus.
7. Ben: Shouldn’t this be equal? Like w minus y? Because if it’s only water evaporating, right? So does that look right?
Ben: Because if 99% is the amount of water [shading a diagram with his pencil] and some of the water evaporates, so there is a percent change here [shades again] and then like, this much water evaporates. Then 98% now, you still have —

Figure 108: Ben’s drawing and work on the Evaporation Problem.

Ben: So then, if your subtracting this much water, and there is this much substance, your only subtracting from the water. Right? So .98 times the total minus .98y equals water minus y. And you'll be able to pull this out, because you can solve for w here. So w equals

Callie: So your saying its [unintelligible]

Ben: I'm saying it’s this one right here. See, .98 times t – y. You split it up (....)
Ben: So, this is the original amount of water, okay. And when it evaporates, this amount of water minus this amount that evaporated now is 98% of the total amount minus the y.

Callie: I thought w meant the original amount.

Ben: Uh huh. This is the original amount.

Callie: So what's t then

Ben: The t is total. This is the total thing. This is just the original amount of water. So this is t.

Callie: Oh, I see. Gotcha ….

Callie: Don't we need — we have two equations, I mean two unknowns. Don't we need two equations?

Ben: Uh huh. We do. That's where this one comes in.

In this excerpt, it is clear that Ben was able to access his conceptual understanding of proportions to make appropriate constructions (both equations and diagrams) to make sense of the problem (see Figure 108). Ben chose to use four variables to represent the quantities in the problem. Callie [1, 3, 10, 13, 15, 17 and 18] and Amy [5] repeatedly asked Ben to clarify the meanings of his variables and equations. Ben had devised a strategy for solving the problem, as evidenced by his responses to Callie and Amy [8, 9, 11, 12 and 19], and he framed his explanations in the context of the problem [2, 6, 7, 8, 9 and 12] without prompting from the instructor. As he explained his reasoning, he also revealed some monitoring behavior by asking Callie [4, 7] if it seemed sensible to her.

In the next passage, Ben followed through by executing his plan.
[20] Callie: Oh, okay. So what is your point 2 equation up there?

    What's left? The point 2?

[21] Ben: This one? Point 1 of the total thing is x

[22] Callie: Right

[23] Ben: And then later, it ends up being 2% of this minus the water

    you take out. It’s the same amount.

[24] Callie: Okay. I can see that.

[25] Ben: That’s good because — see, you could solve for t or y

    here. I mean you have a whole bunch of figuring — so 0.02t minus

    0.02y equals x. And you know that x = 0.01t.

[26] Ben: Oh shoot. Oh no, that's right. And that’s how you get rid of

    the x. Now you have two equations in two unknowns. And then

    you just use these two. .....

[27] Ben: Say what??

[28] Amy: It’s a y in there, right?

[29] Ben: 50%. You take out half of it. Wait, 2 times 5 — you take

    out half.

He executed his calculations correctly, and paused to evaluate his progress so far [26].

Although he was surprised at his result [27], upon reflection he decided that his answer
did indeed make sense [29]. He explained it to his group this way:

    Because then you — this actually doubled in size from the original. Does

    that make sense? So now, instead of 1%, you've got 2%. It’s 1% of 100%,

    so 2% of the 50%. Does that make sense? So, the algebra works out. You
end up taking half of the entire thing out. So you get 2%. Half of the water evaporates. Not half of the water — half of the total volume

![Diagram of the Evaporation Problem](image)

*Figure 109*: Ben’s parameterization of the Evaporation Problem.

For his extension, Ben first parameterized the problem (see Figure 109), and in his explorations, he chose to create a function for the final concentration: 

\[ Y = \frac{TX}{(T-R)} \]

*Figure 110*: Ben’s exploration of the Evaporation Problem.
He arrived at this decision after experimenting with various solutions and noticing how the concentration of the substance varied with the amount of water that evaporated. This suggests that he went through a process of creating and then testing various cases. He then graphed his function and used it to explain the counter-intuitive nature of the solution of the original problem. In Figure 110, he explained why changing the concentration of the substance by 1% (from 1% to 2%) resulted in the loss of ½ the water.

The isomorphic problem he submitted was as follows:

Suppose aliens landed on earth (100,000,000 aliens). The aliens wanted to ship out enough humans so that the aliens made up half the population. What percent of the humans would have to be removed if the current population were 6,375,882,069 humans?

His assertion that this problem was isomorphic to the Evaporation Problem was correct, although he neglected to construct a proof of this assertion.

While solving the Evaporation Problem, Ben was observed engaging in all the phases of the problem solving cycle described by Carlson and Bloom (2005). His extension showed that he considered and tested a variety of cases to assist him in his attempts to make sense of the parameterized version of the problem. These explorations led him to construct an explanation for the unexpected solution to the original problem.

The second analysis assigned during this period was the Completing the Square Problem (Figure 111).
For the function \( f(x) = 6x^2 + 5x - 7 \), complete the square to name the vertex of the function as well as the \( x \) intercepts.

*Figure 111:* The Completing the Square Problem.

By parametrizing the problem, Ben correctly derived the quadratic formula and the general formula for the vertex of a parabola. In his extension, he chose to explore the discriminant and the coefficients from quadratic equations. He wrote the following explanation of his choice:

One of the most powerful parts of this equation is the discriminant \((b^2 - 4ac)\). This tells us how many roots we have. How does this relate to the coefficients and completing the square? It seems logical to explore this because a form of the discriminant appears in the formula for the vertex.
Ben’s explorations (Figure 112) were restricted to validating the properties of the quadratic function that are customarily learned in high school. When presenting his extension to the class, he admitted that he used the exercise to make sense of properties he already knew.

One thing that struck my curiosity relates to the quadratic formula, which is the discriminant and how the discriminant predicts, like, how many solutions you are going to have. And it all seemed very magical to me until I did the extended analysis.

In summarizing his explorations, he wrote the following:
In conclusion while explaining quadratic equations; you can see how the different representations relate and how each representation provides interesting information. Exploring these problems makes quadratic formulas more concrete and predictable.

This reflection suggests that what Ben valued most in his experience with the Completing the Square extension, were the connections he was able to make between the various representations of quadratics.

Ben’s work with extended analyses proved to be uneven during this phase of the semester. When he solved the Evaporation Problem during class, he oriented himself to the problem, developed a strategy for solving the problem, executed the plan, and verified his solution. He also displayed self monitoring as he worked through the problem and explained it to his group. In short, his exhibited problem solving behaviors were similar to those of mathematicians as observed by Carlson and Bloom (2005). Additionally, his exploration appeared to be motivated by genuine curiosity, and it led him to a mathematical explanation of the problem’s surprising answer. In contrast, when extending the Completing the Square Problem, he seemed to be satisfied revisiting and validating aspects of the function he was already familiar with.

Summary of sessions 6-15

During this phase of the course, some changes in Ben’s problem solving behaviors became evident. His assignments revealed that he took fewer shortcuts than the first phase of the class, appeared to think through the problems before constructing his solutions, and employed a wider variety of heuristics than previously observed. In the group problem solving sessions, Ben began to articulate conjectures. His class work also
revealed the emergence of a level of monitoring and mathematical integrity that was not observed before. This was especially evident in his work with the Triangle Problem, in which he repeatedly expressed the need to prove that finding the centroid would be a useful sub-goal for the solution to that problem. While working the Evaporation Problem, Ben articulated all the phases of the problem solving cycle (Carlson & Bloom, 2005), as he explained his reasoning to others in the group. Even though his extension of the Completing the Square Problem did not reveal any mathematical discovery, his extension of the Evaporation Problem was quite thoughtful and provided him with explanatory power about the original problem.

These data suggest that Ben had adopted some of the problem solving dispositions of experienced problem solvers described by Carlson and Bloom (2005), particularly in the areas of conjecturing, planning, and monitoring.

**Final Phases: Sessions 16-22**

During class sessions 16 through 22, Ben's problem solving behavior continued to evolve. In this segment of the instructional sequence, Ben’s ability to clearly articulate solution strategies was evident. He also displayed confidence, curiosity as well as the tendency to reflect upon the sensibility of solutions. His work on the problem sets, his work on class based problems, his interactions with class mates and group mates, and his extended analyses are discussed in the following sections. His final project is also discussed.

**Problem sets**

In general, Ben’s work on the problem sets was excellent. With few exceptions, his work was complete, well thought-out, and well explained.
Ben displayed confidence in his ability to construct viable solutions paths. For example, consider the problem shown in Figure 113.

7.) In Trapezoid $ABCD$ with bases $AB$ and $CD$, we have $AB = 52$, $BC = 12$, $CD = 39$ and $DA = 5$. Find the area of $ABCD$.

![Figure 113: Problem Set 12, #7.](image)

In his solution (Figure 114) he admitted that a formula for the area of a trapezoid would be helpful, but then expressed his confidence that he could still solve the problem, and derived the formula he needed using his geometric understanding of the relationships involved.
Additionally, he continued to show that he could be quite innovative when devising solutions. When solving the Teenager Problem (Figure 115), he employed his knowledge of computer programming to construct an algorithm (Figure 116) that allowed his graphing calculator to solve the problem. When he wrote, “Otherwise you have to approach it through inspection. You could just divide out the ages and use trial and error” he revealed that he considered other solution strategies.

1.) The product of the ages of a group of teenagers is 10,584,000. Find the number of teenagers in the group and the sum of their ages.

Ben demonstrated his ability to construct logical and convincing proofs when required in Problem Sets 13 and 14. Recall that in the first two phases of the semester he tended to stop short of constructing a proof even when explicitly asked.
There were a few exceptions. When solving the problem in Figure 117, he was unable to devise a strategy to determine the relative sizes of $e^\pi$ and $\pi^e$ without the aid of a calculator (Figure 118).

6.) Without using a calculator, rank the following from greatest to least: $e^e, \pi^\pi, e^\pi, \pi^e$ and explain your reasoning.

Like Amy, he used approximations to justify his decision that $e^\pi$ was larger. His approximations were less crude than those Amy chose, leading him to a correct
determination. However, he used numerical calculations (as opposed to logical arguments) to justify his reasoning.

Figure 118: Ben’s work on Problem Set 11, #6.

The passage below is from his presentation of his solution to the class.

[1] Inst: So the smallest and the largest fall out pretty fast.

[2] Ben: Yeah. So that leaves the middles ones and that’s $e^\pi$ and $\pi^e$, and those are kind of difficult. And then Callie helped me kind of understand that the numbers are pretty close to the bases, or, ah, are pretty similar to the exponents, so the larger exponent has the larger effect. So $e^\pi$ is greater than $\pi^e$.


In [2], Ben tried a different argument than the one submitted in his written work, but this justification was not convincing either.

The other two examples of where Ben failed to construct a solution required him to make sense of the context in which the sine function was used. Ben did not attempt the problems shown in Figure 119 and Figure 120. It is unclear whether lack of conceptual understanding, lack of confidence in his knowledge of trigonometry, or some other factor that led him to submit incomplete problem sets.

5.) The altitudes of a parallelogram have lengths 8 and 10 and intersect at an angle whose sine is $\frac{1}{4}$. Compute the area of the parallelogram.

Figure 119: Problem Set 13, #5.
5.) If \( \theta \) is an acute angle and \( \sin 2\theta = a \), find \( \sin \theta + \cos \theta \).

*Figure 120: Problem Set 15, #5.*

During this segment of the semester Ben’s work on the problem sets revealed that his problem solving behaviors and dispositions were continuing to evolve. As in the previous phase, his work was well executed and well explained. He again showed his ability to construct interesting solutions to problems. Recall that in the first segment of the instructional sequence, Ben tended to follow the first idea that came to mind. There was no evidence of that behavior during this segment of the semester, suggesting that he had been reflecting on the efficiency and correctness of his solution paths. He constructed valid mathematical proofs when they were required and expressed confidence in his ability to find solutions. What was different, and surprising, was that he turned in incomplete assignments, suggesting he had less persistence than demonstrated earlier in the semester.

*Class work*

Ben continued to exhibit monitoring behavior when working on problems in class. For example, when working on the System of Equations Problem in Figure 121, the group found their work so far had been unproductive. In the passage below, Ben suggested a different strategy.

Solve the system of equations: 
\[
\frac{xy}{x+y} = \frac{1}{2}; \quad \frac{xz}{x+z} = -\frac{1}{3}; \quad \frac{yz}{y+z} = \frac{1}{7}
\]

*Figure 121: System of Equations Problem.*
[1] Ben: The reason you can't just say $xy$ equals 1, and $xy$ equals is 2 is because $xy$ can equal 2, and this can be 4. You see what I'm saying?


[3] Ben: So if you put a variable in front of it, then you can do it. But now we have 15 variables.

Figure 122: Ben’s work on the System of Equations Problem.

In [1], Ben reflected on their progress, pointing out what he saw as errors in their reasoning. He went on to suggest a way to correct that error in [3], but then rejected this idea because it made the problem more complex.

When working on the Round Trip Problem (Figure 123) Ben solved the problem quickly, and he was able to explain to Amy and Callie his reasoning about the problem.

Suppose that on a round trip, you travel at 30 mph on the way out and at 60 mph on the way back. What was your average speed?

Figure 123: The Round Trip Problem.
[1] Ben: 40 miles per hour. . . .

[2] Amy: We don't know how many miles it is.

[3] Ben: Speed 1 and speed 2. And then so — okay, you set these up.
   
   So, you know, \( t_1 \cdot t_2 \) equals this, just by cross multiply. \( t_2 \).


[5] Ben: And then we wanted total distance over total time, and total distance is —

[\textit{[later]}]

[6] Callie: So I should — I'm letting my distance vary here, and I shouldn't.

[7] Ben: Well, I mean — I mean, this is going to be the same. I mean, you should be able to come out with the same thing.

[8] Callie: But if my distance is \( 30t \), then I have a \( t \), and it's not going to cancel that.


[10] Callie: See what I mean?


[12] Callie: See, my distance varies. My distance is a function of time right now, and that's not right because distance is fixed, like you just said.

[13] Ben: Well, no. Yeah, but see these are still going to be equal. Just — that's just how it goes. But these times are different though.

[14] Callie: Well, I could just say \( 2d \).
Ben: Your times are different. That's all.

Callie: Right.

In this excerpt, there is evidence of Ben constructing and organizing the relevant information [3]. He then establishes two sub-goals, to find the total distance and total time [5] and helped Amy [2] and Callie [7-17] understand that in the problem, the distance was fixed, not variable as Callie had first thought. His fluency with computational approaches is also evident [13-16] Ben’s insights helped guide the group to a correct and efficient solution.

This was not always the case. When working on the Blackboard Problem (Figure 124), Ben recognized that the number of elements in the sequence would be one more than a multiple of 17.

A set of consecutive integers beginning with 1 is written on a blackboard. One number is erased. The arithmetic mean of the remaining numbers is $\frac{357}{17}$. What number was erased?

Figure 124: The Blackboard Problem.

Ben: Does that mean it's over $n + 1$? Or $n - 1$ mean, is it erased and then take the arithmetic mean? So it's $n$ over minus 1. So this could be 18. Or it's something —

Callie: Oh, I wasn't thinking of that.

Ben: 17 something —

Callie: Multiple of 17.

Ben: Plus 1.

Amy: Plus 1.
[7] Ben: So we could just go back like — if this is the mean, 35, then it's around 70. Okay.

In this passage, Ben recognized that an important sub-goal for the problem would be to establish the number of terms in the sequence. His mathematical reasoning and mathematical intuition lead him to the conjecture that the number of terms would be close to 70. (There are actually 69 terms in this sequence.) He introduced the strategy of looking for multiples of 17 to narrow down the choices. Had the group followed this solution path, they might have arrived at a solution that was both correct and efficient. Instead, as discussed in the previous chapter, Amy and Callie ignored his strategy and used guess and check strategies to find the answer.

There were times when Ben appeared to be distracted rather than engaged in the group work. He appeared to lose interest in the Blackboard Problem when the rest of the group decided to solve the problem using guess and check. When solving the Linear Factors Problem (Figure 125), he stared into space as Callie and Amy discovered the pattern required to solve the problem.

1.) For what integers $n$ between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients?

2.) Find an expression for the problem above that will find all integers $n$ that will make $x^2 + x - n$ factor into the product of two linear factors with integer coefficients.

*Figure 125: The Linear Factors Problem.*

The next statements show Ben’s interactions with his group over the course of 20 minutes, as Amy and Callie solved the Linear Factors Problem.

[2] Ben: Do we have to factor them?


[4] Ben: So, it has to be the product of two linear factors.


[6] Ben: Which one is an integer? Does it matter?

[7] Ben: Two linear factors. What—are linear factors again?

These statements suggest that Ben stayed remained in the first phase of the problem solving cycle by continually returning to trying to make sense of the problem statement. It is unclear whether Ben lacked the resources to solve the problem or if this problem failed to hold his interest. It appeared that he was not engaged in solving the problem, and he eventually let Amy and Callie complete the solution for him.

Ben’s grasp of mathematical concepts such as average speed and arithmetic mean were productive contributions to the group problem solving sessions during this phase of the semester. He was able to establish specific sub-goals for the solution process and was observed reflecting on the progress of the group, the products they produced and the reasonableness or correctness of both. Although Ben continued to monitor both his work and the progress of the group, at times it appeared he was not engaged in the work at hand.
Extended Analysis Tasks

The class completed a two-part extended analysis during this time, as well as a final independent extension which will be discussed in a separate section. The first part of the extension of the Box Problem is shown in Figure 126.

Recall the **Box Problem** solved in class:

*Given a sheet of 8.5 x 11 paper, your group wants to construct an open topped box by cutting squares out of the corners. You want to make a box with the greatest possible volume. What are the dimensions of such a box?*

Suppose the rectangular paper has length \(L\) and width \(W\) with square cut out with side \(x\).

1) Find \(V(x,L)\) and then find the \(x(L)\) (that is, \(x\) in terms of \(L\)) that produces the maximum volume.

2) Graph \(x(L)\) and interpret your results in the context of the problem.

3) Locate the points on the graph that represent the original problem and the square (The 8 ½ x 11 sheet and the 11 x 11 sheet.)

*Figure 126: The first extension of the Box Problem.*

Unlike Amy, Ben was able to reason about the existence of two solutions for \(x\) and to logically determine which function represented the maximizing function (See Figure 127).
He then correctly graphed the function. He reflected on the sensibility of his statement when he wrote, “Hmm, so no matter how big \( L \) gets the \( x \) value that maximizes volume remains at about 0.25. So the amount taken out is never more than a fourth of the default side.” His explanation reveals that he was able to keep track of the meanings of the various variables as well as interpret their meaning in the context of the problem.

Because so few students in the class were able to interpret \( x(L) \) as Ben did, a second extension, shown in Figure 128, was assigned to support the other students’ analysis of the Box Problem.
1.) If the critical points for the function \( V = x(1 - 2x)(L - 2x) \) are 
\[
x(L) = \frac{L + 1 \pm \sqrt{L^2 - L + 1}}{6},
\]
determine which is the maximum and which is the minimum. (Recall that \( V \) is a cubic function) Explain your reasoning.

2.) What does the graph of \( x(L)_{\text{max}} \) tell us about the problem?

3.) Find \( \lim_{L \to \infty} x(L)_{\text{max}} \) and interpret the meaning of that limit.

4.) Suppose I have a fixed length of fence that I want to use to create a rectangular shaped pen along the side of a barn. If I call the fixed length “1”, what should the dimensions of the pen be in order to create a pen with maximum area?

5.) How does the problem in #4 relate to the Box Problem?

6.) Consider the problems we have already solved. (The 8 ½ x 11 sheet and the 11 x 11) and compare the area of the folded up region with the base of the box. Can we generalize the finding to the 1 x L sheet? Convince me one way or the other.

Figure 128: The second extension of the Box Problem.

Ben was very engaged during the class discussion of this second extension. In the following passage, Ben tried to explain the behavior of the graph of the maximizing function and his interpretation of the limit of that graph.

[1] Inst: Yeah. A little change in \( x \) has a much more dramatic change — I mean, a little change in the length has a much more dramatic change in \( L \). Alright. Our graph is steeper in the beginning there.

Anything else?

[2] Ben: It’s like, I don’t know. Like you go less — when it's point five, you go any more or less than this, you just grabbed it. (See
Figure 129 for gesture) It would show — I don't know. You go a little bit more than a fourth and then you grabbed it. It would start to get less.

![Image of Ben gesturing]

**Figure 129:** Ben gestures to show folding up the sides of the box.


[4] Ben: Because of the matter of the $L$ changing. It isn't that great when you get up to that.

[5] Inst: Yeah. Basically, the side that's one, alright, is going to be the limiting factor because that's the “1” that you can't take more than — that's going to tell how much you can take out and give you your domain. Just kind of what Ben was talking about there.

In this passage, Ben attempted to explain how the side of the box that measures 1 constrains the volume as $L$ increases [2, 4]. The instructor then reformulated his statement [3, 5] so that other students might understand it better.

In the next excerpt, Ben described what he saw as the connection between the Fencing Problem and the Box extension.
Okay. Four is a nice problem. We've done that a lot, right? So we get, again, that if this is one, this is a fourth, this is a half, and this is a fourth. So let's see if we talk about what that had to do with the Box Problem. . . Alright, we use similar methods. It's more than that. Okay. So Lena says both problems are needing find the maximum. All right, that's true. Anyone take it a little further? Ben.

Ben: That's like we're thinking — you know, like when I do the calculations for the Fence Problem, this is like $x, x, y$. And then when — if you were to like kind of fit this into the other problem, into the Volume Problem, here like — I draw like that. And then our Volume Problem, this equals 1, so like this is $x$, this is $x$, this is $x$. I mean, you know that $2x + y = 1$, like that. Then you can use — when you maximize this, you find that it's — what is it, a forth, right? Which is kind of the same thing like in the Volume problem. You know that when you maximize, this $x$ right here has to be a forth. And for the Volume Problem, and when you maximize this, it's the same thing. It's just kind of related if you draw—
Figure 130: Ben’s work on #4 from the second extension of the Box Problem.

In this passage, Ben revealed [2] that he saw that the Fencing Problem was isomorphic to the cross section of the Box Problem. His written work (Figure 130) provides further evidence that Ben found the mathematical structure that connected the two problems. Both passages revealed Ben’s continual efforts to make sense of the mathematical expressions he was confronted with in this exercise.

When asked to reflect on the experience of extending the Box Problem, Ben wrote the following (Figure 131).
The box problem helped cement a few ideas into my brain. The first was the idea of modeling physical problems through mathematical expressions. It is nice to be able to quantify real life situations through mathematics. Secondly, it helped me to be critical about graphs of equations that reflect real life situations. You must understand the limits of the problem and where the useful information lies. I also had no idea that the amount you remove (x) approaches a limit when maximizing volume in relation to a continually growing side. It is hard to conceptualize a point where one side would not change because the other length is always getting longer.

Through the further analyzation of the problem I discovered the relationship of the fence problem to the box problem that the fence equation relates directly to the standard one length side of the box problem. $2x + y = 1$ was the equation that came up in both. It was also interesting to see how maximizing the area of the folded side occurs when $x$ is equal to the limit of the maximized box problem. Interesting.

I felt I had all the necessary tools to understand the problem but introducing the fence problem was a great way to link different mathematical ideas.

*Figure 131:* Ben’s reflection on his experience extending the Box Problem.

In this reflection, Ben not only provided a critical evaluation of the challenges he experienced while solving this problem, he expressed excitement about discovering the relationships between various mathematical ideas embodied in the task.

Ben’s work on the two Box extensions revealed curiosity and excitement regarding the discoveries he was able to make. He demonstrated his ability to interpret the functions involved in the context of the problem statements even though $x$ in this case was a dependent variable instead of an independent variable. In addition, he was able to connect the mathematical structure of the Fencing Problem to the Box Problem. He displayed perseverance not only by completing both assignments (unlike Amy) but also by engaging in sense-making throughout his work on both extensions. Additionally, his contributions to class discussions could be seen as signs of mathematical intimacy. In his written reflection about the problem he focused on how he was able to “link different mathematical ideas”.
Final project

For a final project, the students in the class were asked to find a problem from a high school mathematics text or other high school level resources, and extend it. Ben chose the pre-calculus problem in Figure 132.

Convert $\theta = 65^\circ$, $r = 4$ from polar coordinates to rectangular coordinates.

Figure 132: The problem selected for Ben’s final project.

After solving the initial problem correctly, he took the opportunity to explore connections between expressions in both systems. Figure 133 shows one such exploration. Although it is not clear whether these are new discoveries for him, or if he was simply validating connections he already knew, he did connect the use of polar coordinates to programming video games.

Most students fail to see any practicality to a complex way of viewing the location of a point on the coordinate plane, so here is one application.

VIDEO GAMES!!! Many video games make use of this in order to do basic positioning.
He then described a program he wrote for a graphing calculator which would create a fishing boat and move it around the screen. The program made direct use of the discoveries he outlined in the earlier part of the extension.

![Diagram](image.png)

*Figure 133: Example on an exploration in Ben’s final project.*

These explorations prompted him to draw the conclusions described in Figure 134.
In conclusion:

Having a good understanding of polar coordinates and understanding the interaction and relation between rectangular and polar coordinate systems, you can enhance your visualization, mathematical and critical thinking skills. Trigonometry is also key in the conversions and identifying the role that the angle and radius play are important in really grasping polar coordinates…

It is also good to explore this area because its exploration involves a lot of reversibility and path independence.

Figure 134: Ben’s statement of conclusions drawn from his final project.

His final extension and these conclusions suggest that for Ben, understanding both polar and rectangular coordinates meant understanding the connections between them and how the various factors such as radius and angle were expressed in both systems.

When asked to reflect on his experiences with all the extended analyses, he expressed the following;

Extended analyses are probably the most effective way of teaching mathematical relationships and concepts. It is a method that used self discovery and critical thinking, which are two of the most profound and long lasting ways of learning material. Although not entirely easy, they are actually fun to do because of the discoveries and connections that are made in the process.

He found the Completing the Square extension especially influential. He wrote

Other problems like the quadratic equation problem gave amazing insight into the derivation of a fairly complex formula that always seems mystical… I believe I performed well in the competing the square activity
that led to discoveries about the quadratic formula, the vertex and relationships to the graphs of the quadratic equations. It all flowed naturally and I believe that it was fairly easy to see the connections.

Ben’s final extension and his subsequent reflections reveal that he found the process of making connections was the most valuable aspect of the experience for him.

*Summary of sessions 16-22*

Ben’s problem solving abilities and mathematical dispositions continued to evolve during this last phase of the instructional sequence. His ability to clearly articulate solution strategies was evident. He also displayed confidence, curiosity as well as the tendency to reflect upon the sensibility of solutions. His strong mathematical conceptions afforded him the ability to construct well executed and well explained solution paths. This was especially evident in his work with the two Box extensions. In his final project and various written reflections, Ben expressed that he found that connecting across mathematical ideas was especially important for him.

*After Instruction*

This section describes Ben’s performance in the post-instruction measures. These measures include the results of a posttest comprised of the same NAEP items as the pretest, a VAMS survey, and a second task-based interview. Additionally, these results will be compared to the pre-instruction measures for evidence of change.

*Results from the Posttest*

Ben completed the posttest at the end of the instructional sequence. Even though it was identical to the pretest (the complete instrument appears in Appendix A), his
responses showed only a modest improvement. Ben answered 28 of the 30 questions
correctly, indicating a net gain of one question.

Table 18

*Ben’s Gains on Posttest Items*

<table>
<thead>
<tr>
<th>Item number</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>A rectangular pool 24 feet long, 8 feet wide, and 4 feet deep is filled with water. Water is leaking from the pool at the rate of 0.4 cubic foot per minute. At this rate, how many hours will it take for the water level to drop 1 foot?</td>
</tr>
<tr>
<td>16</td>
<td>A certain company keeps a list of 50 employees and their annual salaries. When the salary of the very highly paid president is added to the list, which of the following statistics is most likely to be approximately the same or nearly the same for the original list and the new list?</td>
</tr>
<tr>
<td>18</td>
<td>![Diagram of three cylinders]</td>
</tr>
</tbody>
</table>

In the figures above, the radius and height of each right circular cylinder are given. If \( w, x, \) and \( y \) represent the respective volumes of the cylinders, which of the following statements is true?

\[ n=3 \]

Although he made gains by correctly solving three items he had missed before (Table 18), he also gave incorrect responses on two items he had answered correctly on the pretest (Table 19), yielding a net gain of 3.3%. 
Table 19

Ben's Incorrect Responses on Posttest

<table>
<thead>
<tr>
<th>Item number</th>
<th>Test item</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>The population of the United States is approximately 250 million, and the national debt is approximately 4 trillion dollars. If this debt were divided equally among the population, what would be the debt, in dollars, per person?</td>
</tr>
<tr>
<td>11</td>
<td>Suppose $4r = 3s = 10t$, where $r$, $s$, and $t$ are positive integers. What is the sum of the least values of $r$, $s$, and $t$ for which this equality is true?</td>
</tr>
</tbody>
</table>

It is interesting to note that both of the problems that he answered correctly on the pretest but incorrectly on the posttest were classified in the content area of Number Sense. He demonstrated gains in the areas of Measurement and Data Analysis, Statistics, and Probability (Table 20). Ben had responded incorrectly to half of the Measurement questions on the pretest, whereas he answered all of the Measurement questions correctly making this the content area where he demonstrated the greatest gains on this instrument.
Table 20

*Ben’s Gains by Content Area*

<table>
<thead>
<tr>
<th>Content area</th>
<th>Number of items</th>
<th>Number of correct items on pretest</th>
<th>Number of correct items on posttest</th>
<th>Percent change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry and spatial sense</td>
<td>n=4</td>
<td>4</td>
<td>4</td>
<td>0%</td>
</tr>
<tr>
<td>Number sense, properties, and operations</td>
<td>n=8</td>
<td>8</td>
<td>6</td>
<td>-25%</td>
</tr>
<tr>
<td>Measurement</td>
<td>n=4</td>
<td>2</td>
<td>4</td>
<td>+50%</td>
</tr>
<tr>
<td>Algebra and functions</td>
<td>n=10</td>
<td>10</td>
<td>10</td>
<td>0%</td>
</tr>
<tr>
<td>Data analysis, statistics, and probability</td>
<td>n=4</td>
<td>3</td>
<td>4</td>
<td>+25%</td>
</tr>
<tr>
<td><strong>totals</strong></td>
<td>30 total</td>
<td>27</td>
<td>28</td>
<td>+3.3%</td>
</tr>
</tbody>
</table>

When his results are organized by ability type, his gains on the posttest were all characterized as Problem Solving, while his losses were all categorized as Conceptual
Understanding (Table 21). The losses in the category of Conceptual Understanding are somewhat perplexing as Ben’s conceptual understanding of mathematics was shown to be strong during the instructional sequence.

Table 21

**Ben's Gains by Ability Type**

<table>
<thead>
<tr>
<th>Mathematical Ability</th>
<th>Number of items</th>
<th>Number of correct items on pretest</th>
<th>Number of correct items on posttest</th>
<th>Percent change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem Solving</td>
<td>13</td>
<td>11</td>
<td>13</td>
<td>+15%</td>
</tr>
<tr>
<td>Conceptual Understanding</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>-10%</td>
</tr>
<tr>
<td>Procedural Understanding</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>0%</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>30</strong></td>
<td><strong>27</strong></td>
<td><strong>28</strong></td>
<td><strong>+3.3% overall</strong></td>
</tr>
</tbody>
</table>

Ben’s performance on the posttest showed an overall gain of 3.3%. He made the greatest gains in the content area, Measurement and the ability type, Problem Solving. His score’s rank, when compared to other posttest scores in the class, remains the same with two students attaining scores higher than his.

**Views About Mathematics Results**

The results of Ben’s post-instruction VAMS revealed positive shifts toward expert views about the practice of mathematics. Although Ben’s pre-instruction VAMS had categorized him as holding expert views already, his post-instruction survey (Table 22)
showed a total of four more of his responses that could be characterized as expert, and fewer of his responses were characterized as espousing folk or mixed views.

Table 22

*Comparison of Ben’s Pre- and Post- VAMS Results*

<table>
<thead>
<tr>
<th>Survey</th>
<th>Number of expert responses</th>
<th>Number of mixed responses</th>
<th>Number of folk responses</th>
<th>View category</th>
</tr>
</thead>
<tbody>
<tr>
<td>pre</td>
<td>13</td>
<td>6</td>
<td>3</td>
<td>expert</td>
</tr>
<tr>
<td>post</td>
<td>17</td>
<td>3</td>
<td>2</td>
<td>expert</td>
</tr>
</tbody>
</table>

Table 23 shows the survey items that showed change. The complete survey may be found in Appendix B.
Table 23

*Items Indicating the Greatest Gains on VAMS*

<table>
<thead>
<tr>
<th>VAMS items showing change</th>
<th>Ben’s pretest response</th>
<th>Ben’s posttest response</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Towards “Only (a)”</td>
<td>Equally (a) &amp; (b)</td>
<td>Towards “Only (b)”</td>
</tr>
</tbody>
</table>

4. My score on mathematics exams is a measure of how well:
   
   (a) I understand the covered material.
   
   (b) I can do things the way they are done by the teacher or in some course materials.

   [Not categorized by Carlson et al.]

5. For me, doing well in mathematics courses depends on:

   (a) how much effort I put into studying.

   (b) how well the teacher explains things in class.

   [Expert View: options 1-4; Mixed View: option 5; Folk View: options 6-7]

10. Mathematical formulas:

    (a) express meaningful relationships among variables.

    (b) provide ways to get numerical answers to problems.

    [Expert View: options 1-3; Mixed View: option 4; Folk View: options 5-7]
<table>
<thead>
<tr>
<th>VAMS items showing change</th>
<th>Ben’s pretest response</th>
<th>Ben’s posttest response</th>
</tr>
</thead>
<tbody>
<tr>
<td>19. Collecting and graphing real world data is useful for:</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>(a) determining patterns and making general predictions.</td>
<td>[Expert View: options 1-3; Mixed View: options 4-5; Folk View: options 6-7]</td>
<td></td>
</tr>
<tr>
<td>(b) obtaining numerical answers to specific problems.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20. For me, making unsuccessful attempts when solving a mathematics problem is:</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>(a) a natural part of my pursuit of a solution to the problem.</td>
<td>[Expert View: options 1-2; Mixed View: option 3; Folk View: options 4-7]</td>
<td></td>
</tr>
<tr>
<td>(b) an indication of my incompetence in mathematics.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24. In order to prove a mathematical theorem one must:</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>(a) produce evidence from the physical world.</td>
<td>[Expert View: options 6-7; Mixed View: options 4-5; Folk View: options 1-3]</td>
<td></td>
</tr>
<tr>
<td>(b) provide a logically sound argument.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ben’s VAMS responses to #10, #19, and #24 suggest a shift towards a view that the study of mathematics is more about patterns and relationships and less about ways to...
find numerical solutions to problems. His response to #20 suggests that he might be
developing a more mature view of himself as a problem solver by acknowledging that
unsuccessful solution attempts are part of the problem solving process. Items #4 and #5
are interesting because they deal with the roles and responsibilities of teachers and
students in the class. The movement in these items reveals that for Ben, successful
teaching and learning of mathematics may no longer be just about how well a teacher
explains things or how well a student replicates a teacher’s work. This shift is interesting
because these ideas were never formally addressed during the course, suggesting that the
experience of the way the course was conducted may account for the change. It is worth
noting that Amy demonstrated shifts on the exact same survey items, suggesting that
these may be attitudes that were fostered by the instructional sequence

_The Post-Instruction Interview_

The next pages report the results of a second semi-structured interview with Ben
conducted at the conclusion of the instructional sequence. The interview, held
immediately after class, lasted approximately 60 minutes and was conducted by me (the
researcher and instructor). Ben was asked a variety of questions about his experiences
during the class (refer to Figure 8), and was asked to solve three mathematics problems
(refer to Figure 9) using a talk aloud protocol.

_Responses to the general questions_

When asked if his knowledge base had changed during the semester, Ben stated
that he felt he had grown mathematically. When asked if his problem solving abilities had
changed, he replied with the following:
Sometimes — like I remember in the beginning when we had the first interview, I would be stubborn and continue with one kind of assumption for a long time. And then after we read that article and doing so many problems and seeing that if you explore a couple of different areas, you can kind of tell which way might work better or might just totally disprove your first assumption. So I'm less stubborn to stay on one and like be quicker to explore other options, if I get stuck.

In this passage, Ben recognized that his early tendency to follow his first thoughts was a weakness and that he felt he was now more likely to explore other strategies.

When asked what activities were especially influential or revealing, he said he was most influenced by working on extended analyses.

Extended analysis problems for sure. For revealing all kinds of math and definitely influenced the way you looked at problems. Even, like, parameterization helped you, kind of, helped you predict. If you just looked at just numbers, you can find the answer, but by doing the extended analysis and the parameterization you can kind of look at variables and understand how they interact or kind of predict your answer before you, before you actually solve it, and that’s nice.

This statement is aligned with some of the shifts revealed by VAMS in that he expressed interest in mathematical relationships and structure after the instruction sequence and it appears that he is crediting his interaction with extended analyses for this change in his perspective.
When asked about changes in his level of confidence in his understanding of high school topics, he stated the following:

Yeah. I think I’m more confident. Sometimes when kids will come up to you with homework problems, you’re not sure what section they’re working on, you’re not sure what formulas they are using so – and it’s just like when you give us problems in class. Like we don’t know – usually you get a problem, it’s in a section and you know I’m going to be using what I just learned. So it’s nice to have all these problems in our class, so that when they come in to me with problems – I don’t even know, I don’t have to know what they are studying. I can just kind of feel my way around it and then kind of accumulate what I might need, and then go from there. So, that helps.

Ben had also expressed confidence in the pre-instruction interview, even though he admitted that he might need to “brush up” on some topics. In the above statement, he revealed that he now felt he had the confidence to help students with all topics. He went on to express how the instructional sequence influenced his field work in a local high school.

[Be]ing in the high school with field experience, you kind of think more behind the scenes to see what more basic skills you can expose to make sure they have a grasp of that before you like get onto a bigger problem. This statement suggests that he now has a different perspective on the topics he will teach. He went on to say it had influenced the way he viewed teaching in that “if I can't
— they’re not understanding one way, I’m not going to push, you know, maybe a certain
curriculum on them or a certain lesson, maybe change it. That’s probably mostly it.”

At the beginning of the semester, Ben expressed an interest in teaching calculus.
Now when asked what he was most looking forward to teaching, he had a different
perspective.

I want to teach the ones where they get really interested, um, in making
connections. I think you can like, make connections in every math class, I
know you end up making a ton in geometry, and it’s always very
interesting to see the relationships between everything in geometry. I
wouldn’t, I think I would like to have a geometry class. There’s a lot of
like hands-on type stuff you can do with it. I also like calculus a lot
because it’s more like upper level thinking and I think just how you can
tell like the area under a curve from like an equation and get slopes form
an equation, I think that’s pretty fascinating, and I think portraying that
enthusiasm would be pretty easy to, like, the kids that are in the class, you
know, so I think that would be a fun class too.

In this passage, Ben expressed a desire to help students see relationships in mathematics.
Recall that in Ben’s last written reflection, he had revealed that finding relationships and
connections in mathematics held value for him. In this statement, he wishes to share that
with future students.

In summary, Ben’s responses to these questions revealed that he believed that his
understanding of mathematics had been enriched by his experience in the class and that
he experienced improvement in his problem solving abilities and felt even more confident
in the topics that make up the high school mathematics curriculum. In addition, he credited the Extended Analyses Tasks as being especially influential in helping him see the importance of connections and relationships between ideas and topics in mathematics. Additionally, he reported that the experience with the instructional sequence influenced how he thought about teaching and learning mathematics.

**The problem solving session**

Ben was asked to attempt three problems and to share his thinking aloud. The first problem he attempted was the Quarter-circle Problem (Figure 135).

![Quarter-circle Problem](image)

Figure 135: The Quarter-circle Problem.

[1] Ben: Okay. Um, perimeter of the shaded region. So for this it would be nice — well, you could think arc length. I was thinking maybe arc length might be important, but otherwise you know it's a quarter of the circle.

[2] Ben: So you can figure that out without using arc length. What's left is just knowing this and this and that [pointing to segments AP, PR and BR on figure]. And I'm assuming that there's going to be
some kind of formula that can help us figure out this length and this length. Because we have kind of a generality of what this is going to be considering that the whole length is 10. So we can find this right away by dividing the circumference by 4. So $2\pi - 10$, 25 over 4, so that's this. We know that the perimeter of the rectangle is 26. And we're most interested in this side [indicating segment PR] to begin with. Let's see, what else do we know? I wonder —


[4] Ben: I'm trying to think of, like properties of inscribed legs of triangles, but it's a quarter circle, so it's kind of odd. Let's see. Like a relationship between all these — like a length of this side and this side and this side and this side are kind of weird. But you know that this plus this equals that plus that.

[5] Ben: But it's going to be kind of weird to see if the length — I don't know. I'm sure they are. So it would be $BR$ plus $RC$ equals $AP$ plus $PC$. And these both equal 10.

[6] Ben: And you know that $PC$ plus $RC$ — well, let's see, two of those and two of these equals 26. That's always good to have, in terms of two variables, you know, instead of all four of them.

In this passage, Ben oriented himself to the problem space [1, 2] and immediately set a sub-goal of determining the length of the arc [2]. In statements [2] and [4], he appeared to be scanning his knowledge about the figures involved and tried to find useful
relationships [4] and possible plans of action [6]. Following this passage, Ben was able to find all the lengths except the length of \( PR \).

[7] Ben: So I guess that's kind of good to know. I'm wondering if I'm going to go in circles how many variables to add. Plus \( RC \) equals \( BC \) is 10. We really just want to know this \([\text{indicating } PR]\), and if we can find this and this \([\text{indicating } AP \text{ and } BR]\), maybe we can find that, and then maybe these, in terms of the other ones too. So we need to find \( BR \), and we know what \( BR \) is. And \( AP = AP \) equals 10 minus \( PC \). And \( PC \) equals 13 minus \( RC \). \( AP \) equals 10 minus 13 plus \( RC \). So then I have \( AP \). And we need \( PR \), and \( PR \) is — we can use the Pythagorean Theorem. This might get complicated.

[8] Ben: Yeah, yeah. We're looking for \( PR \), and \( PR \) equals \( RC \) squared plus \( AP \) squared. And \( AP \) equals 10 minus \( PC \), so hopefully some stuff will cancel. 10 minus \( PC \). Wait, no. What am I doing? \( AP \) squared, no. \( PC \) squared — 15 minus \( RC \). And this one is using \( RC \) — let's see. Okay, that's good.

[9] Int: So what are you thinking?

[10] Ben: Well, I don't know if want to substitute cause I have this equation, but I want to keep more of one of the variables, and I'll just use \( RC \) as like \( RC \), I suppose. \( RC \) squared —

[11] Ben: Maybe I could — if I need to, I will substitute it later. 13 squared — what's 13 squared, 169 or something?

Ben: It’s kind of nasty. It’s going to be weird.

Ben: So, at this point, seeing how kind of weird it is. I can’t take out a 2 out of here so I’m going to have to use the quadratic formula. Like, I don’t know, I can’t exactly take a square root. I was hoping it’d be a perfect square. It might not, it might not be pretty, so, I don’t know. I’d better think of some other ways to go about this.

Ben: Let’s see. I also know, what do I know? I don’t know. It’s just kind of difficult to see if there is a relationship between, or finding, just basically finding these lengths. Just knowing the 10, um. [long pause]

In this passage, Ben appeared to be making conjectures about possible solution paths [7, 8, 10] and trying to assess where they might lead [8, 11, 14, 15]. Statements such as “that’s good” indicated some monitoring along the way. Ben opted to try another problem when offered the choice, but he did come back to the problem later in the interview.

When he returned to the problem, he started by redrawing the figure (Figure 136) and looking at the relationships involved again.

Ben: That’s good. The perimeter of this is, it’s really – I don’t know. It seems like you should, like the way these points are going to act, like if I were to draw one here, I feel like I should be able to find these, knowing like, these sides. You know what I’m saying? Like it shouldn’t really be a problem, but the fact that it’s given to
you in the form of perimeter, you know what I’m saying? It’s just
going to be a little more difficult. Again, just knowing what’s
given, it’s always good to write down. In fact, I’m going to change
– all these letters are crazy, so we’re going so – I’m going to draw
my own and this’ll be $a$ and $b$ and $c$, $d$ and the other part here, $e$.
And we know, we know this and it’s the perimeter is
$5\pi + a + e + d$ and that’s going to be our goal. And we know that
$2b + 2c$ is going to equal 26, and we know that $a + b = 10$, which
also equals $d + c$. So we have three variables – $a$, $b$, — we have
four variables that are kind of interacting and they are all, I mean
they are all bound by, like what’s going on here.

Figure 136: Ben’s work on the Quarter-circle Problem

Once he labeled his diagram he set up relationships and began executing his plan. In the
next excerpt, he stopped and reflected on where he was in the solution.

[2] Ben: Let’s see. I need – I just need to find $e$, but – okay, so

$5\pi + 7 + e$. Um, $e = \text{the square root of } b \text{ squared plus } c \text{ squared}.

So then I know this, and we can use the fact that $a + d = 7$
somewhere in there. I’ll try this and see, so plugging in the things we found from earlier. . . . Now the problem is finding – if I can get $a^2 + d^2$ into some kind of constant, um, and I think

$$(a + d)^2 = a^2 + 2ad + d^2.$$  

[3] Ben: And that equals 49. So this equals 49 minus $2ad$. So what’s the square root of $109-2ad$? And then, I don’t know, that’s kind of odd. So that’s my $e$. I know it would just be nice if the values for $a$ and $d$, they just – but that 26 is, I don’t know. They’re varying, but it seems they should be bound by that 26 because there is only so – no matter how you draw it, you should be able to predict, like, these lengths because it’s, they’re all unique. Especially for the fact that I know the radius. So if I can figure out how, like each one of these is unique, then I would probably be able to just plug those in later. It would probably be pretty simple, but it’s uncovering that that’s kind of the difficult part.

[long pause]

[4] Ben: Oh, wait a minute. [laughs] So, this is the radius, and that’s 10, so $5\pi + 7 + 10$.

In [3], Ben stopped again to reflect on his progress. He considered the relationship between $a$ and $d$ by saying “they should be bound by that 26 because there is only so – no matter how you draw it, you should be able to predict, like, these lengths.” This reflection
appeared to lead him to see that \( e \) would have the same length as the radius, which was the piece he needed to complete the solution.

Next, Ben solved the Bottle Problem (Figure 137).

Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that is in the bottle

\[\text{Figure 137: The Bottle Problem.}\]

[1] Ben: Okay. So first I’ll label the axes just so you kind of don’t lose sight of what you are doing. We just did time graphs in this algebra class and it’s hard for them to remember if they are doing whatever verses time. Um, you said the amount of water?


[3] Ben: And that’s the height. So, I don’t know, in this case I would think – I’ll just make it like a perfect like cylinder kind of, and then relate it to if you poured in like a cup or something and it came up here, a cup would probably be – this would be like if these were to scale, if these were like compared to each other.

[4] Ben: So --- you see that? This is going to be pretty, pretty much linear. This one is not going to be linear. Um, because one cup up here is going to
make a difference of height – a bigger difference in height than it would right here because it’s going to be dispersed over a wider area. So it starts out, it kind of goes up pretty quick, um, because it’s not very wide, so the skinnier it is the quicker it’s going to go up, so it’s really a function of how skinny it is. So we can just make some marks just to make sure it’s an even amount of water, so it’s going to go up kind of quicker to start. So we’ll just say to here. And then immediately it’s going to start slowing down. So it’s going to be – it’s going to be increasing the whole time because the height always is increasing but it’s not – it’s going to be slowing down for a while. And then it kind of hits an inflection kind of point, and it will start to go back up. So this will kind of correspond to there.

Figure 138: Ben’s graph of the Bottle Problem.

[5] Int: Okay. So the inflection point is where the bottle is at its widest?
[6] Ben: Yeah. And then it will start to fill up faster, because you are putting in the same amount of water, but that amount of water is going to have a bigger effect on the height when it’s skinnier, so it will start to go up to a certain point – to here – which is going to be pretty similar to like my first point. And then, so it’s going to basically be symmetric from here to where ever – one, two – yeah, to like here. So like this part would be symmetric, kind of. In terms of, like, the slopes. And after that, it’s kind of like my cylinder – it’s going to be more linear. It’s not changing from here to here, so, and it’s the same height as here, kind of, it’s the same width as you approach right here, so it’s just going to continue on straight.

In [1], Ben oriented himself to the problem, taking care to clarify the variables. In [3] he articulated his plan of imagining filling the bottle cup by cup. In [4] and [6], Ben engaged in sense-making as he executed his plan. Ben’s understanding of rate-of-change is evident when he determines which portions of the graph will be linear, which will not be linear and where there exists an inflection point.

The last problem he attempted was the Smallest Area Problem (Figure 139)

Of all lines through the point (5,2), which one cuts off the region of smallest area in the first quadrant?

Figure 139: The Smallest Area Problem.
Ben: Okay. Smallest, I need minimum. So you could use calculus to set up, like, area in terms of the different lines. And each one makes a triangle, so you could set up area in terms of the triangle. And each triangle is unique in terms of this point, so what you could do is just move up this way. Start here – I mean you could go this way too – and go up like that. Then you’d be able to find this point in terms of this point and the given, and then set up an area problem. So, if, let’s see. And there are only so many points you can use, so if you have a point down here, it’s going to shoot off into infinity. So it has to – and same in here too – so really, you’re going to start at a certain part. And you can’t have any in here either, because if the point was here, this would also shoot off to infinity. So, you’re kind of bounded, and just like with the other problems where you do your minimum or maximum problems, some of your points won’t work. So you would start at 2 here and 5 here.

Ben: So, when you pick this – we’ll just call it $y$ um, we’ll maximize in terms of $y$ so the area – we’ll use this generic one – this is $y$ and this is $x$ – is going to be $x$ times $y$ over 2. So the area of the triangle and then $y$ is just $y$. So we can use, we’ll just keep $y$ kind of as our variable and find $x$ in terms of $y$. And to do that, we would use this, and really, we would just find the equation of the line because you want to find $x$ and put zero in for $x$ to find $x$. So,
you would use point-slope, and you would have \((0,y)\) for your first point and \((5,2)\) for your second. And your slope is \(2 - y\) over \(5 - 0\). And then point slope formula would give you \(y\) minus whatever – I’ll just use this point because there’s [sic] no variables in it – equals \(2 - y\) over \(5\). Although if you did use this point you would get — you would get zero. No! Because this is a different \(y\). So let’s stick with that – right, okay. I almost mixed up the variables. Um, 2 minus \(y\) over 5 times \(x\) minus 5. So \(y\) equals 2 minus \(y\) over 5 – hold on.

[3] Int: What are you thinking?

[4] Ben: I’m just like, this, \(y\) is kind of like the equation \(y\) from the formula and this other \(y\) is like my variable and when I want to solve for \(x\) I’m going to get two different kind of \(ys\) and I don’t want to confuse them.

[5] Ben: So let me start over. We’ll call this – this started getting confusing – oh, let’s call it \(a\), and this will be \(b\). I just want to make sure I start it fresh. So now these are points \((0,a)\) and \((5,2)\) and we want to find the \(x\) in terms of that and the other point. So…okay. Right, okay. So 2 minus \(a\) over – whoa, wait wait wait. Yeah, so those are our slopes, so now we have \(y\) equals 2 minus \(a\) over 5, and I probably could have just kept on going but I don’t like changing \(ys\). – \(x\) minus 5. So minus 5. So \(y\) equals 2 minus \(a\) over 5 \(x\) minus 2 minus \(a\), and then plus 2. These are going to cancel, so \(y\)
equals 2 minus \( a \) over 5 minus \( a \). And this is our given, so now I have kind of like our \( x \) in terms of \( y \) – kind of. And now, because it’s kind of like a given, \( x \) and \( y \)….  

*Figure 140: Some of Ben’s work on the Smallest Area Problem.*

[6] So, it’s kind of nice because now we have an equation of the line in terms of these two points and these are pretty much, kind of given even though that’s kind of a variable. So to find this in terms of this \( b \) which we need, we just plug in zero for \( a \). Yup, and we get \( y \) – zero in for \( y \) rather – because this \( a \) is kind of like our \( y \) intercept, this is now constant so, you plug in zero over here. And
solving for $x$, we get $x = a \cdot \frac{5}{2-a}$, which is $\frac{5a}{2-a}$. Then we have

our $x$ slash $b$ and area equals $\frac{5a}{2-a}$ times $a$, which is just our

given, over 2. So $\frac{5a^2}{2-a}$, and that’s all over 2, so, put a 2 over

there, and then you would just use, take the derivative of this.

[pause] And the derivative of this – you would use the quotient

rule, which is – I don’t really remember but I think it’s –

In [1], Ben engaged in sense making as he imagined the various lines that could pass
through the point. He began to devise a plan in [2], but in [4] he recognized that he had
made poor choices when naming his variables. This is interesting because in this case
( unlike his behavior during the pre-instruction problem solving session), Ben recognized
and corrected his notational difficulties [5] early on. He began to execute his plan in [6];
however, in [5] he made an arithmetic error when he failed to distribute a negative sign in
his calculations. After finding the equation of the line, he checked his solution.

[7] Oh, yeah it is. So it would be somewhere up here. And you could

plug it in to get your – where’d that equation go – which one was

it? This one. You could plug in 4 here. $\frac{-2}{5}x - 4$, oh wait, what do

we want? Oh we have to solve for $x$. Where’s our $x$ at? Oh, here’s

the $x$. So 4 times 5 over negative 2, which is 20 over negative 2.

Hmmm. Negative 10. That’s kind of weird.
[8] Ben: Wait, what am I doing? My \( a \) is up here, in terms of that line – okay. Yeah, so if it’s 4 here – will it work? This is the equation of the line, with \( a \) equaling – we want the \( a \) to equal 4 and this would be the equation, so when you want —

[9] Int: When \( a = 4 \), what is the equation?

[10] Ben: It’s \(-\frac{2}{5}x - 4\). And then, that’s the equation of the line. And when \( y \) is 4, of, when this is zero. When \( x \) is zero, \( y \) equals negative 4? So, am I missing a sign? Because what should happen, this should be a 4, you see what I’m saying? So there is probably a sign messed up somewhere. I don’t know if it’s that one right there. It’s probably, maybe, I should just go like that. I don’t know. I’d probably have to redo a lot of the math.

In [7] he realized that an \( x \) intercept of negative 10 did not make sense. It appeared that he intuited that the equation should be \( y = -\frac{2}{5}x + 4 \), and he made the change on his written work (see Figure 140). This was the only one of the three problems Ben failed to solve correctly (although he did write down the correct answer).

For each of the problems from the problem solving session, Ben oriented himself to the problem, constructed strategies, and executed them. Ben accessed the appropriate mathematical resources for each problem. When he revisited the Quarter-circle Problem, he was able to find the final piece of the solution be reflecting on the relationships in the problem. His monitoring behavior also alerted him to problems with the variables he
chose in the Smallest Area Problem. He expressed confidence in his solutions and knew he could have found his error on the last problem if he had had more time. The problem solving behaviors Ben revealed in this problem solving session were similar to those observed in more experienced problem solvers (Carlson & Bloom, 2005).

**Summary of Ben’s Transformation**

This section describes the evolution of Ben’s mathematical and problem solving behaviors as he progressed from completing the pre-instruction measures, through the 22 sessions of the instructional sequence, through to completing the post-instruction measures.

Pre-instruction data suggested that Ben possessed a strong mathematical background prior to instruction. His pretest score was ranked high in the class, suggesting that his content knowledge was relatively strong. VAMS revealed that he held what could be considered expert beliefs about the practice of mathematics.

Ben reported that he was confident about his mathematical knowledge; he said he was persistent in his efforts to solve problems but admitted that he rarely planned out a solution path. However, in the problem solving session he was not always persistent or particularly confident. Ben was able to access key mathematical ideas needed to solve the problems. He was also observed engaging in what appeared to be monitoring behavior, but such monitoring did not alert him to errors in his reasoning, notational choices or computations.

In general, the pre-instruction data indicated that Ben was one of the strongest students in the class at the beginning of the semester.
Ben's work in this early phase of the semester was not surprising considering his performance on the pre-instruction measures; he was able to access key mathematical concepts when solving problems and appeared to be fairly confident and persistent in his efforts. However, analysis of data from the first five sessions revealed that Ben's work, both individually and while interacting with peers, could be uneven. Sometimes he committed himself to a plan of action with little reflection on products or processes of his work; other times, he displayed fluency with a wide repertoire of computational approaches, submitting elegant solutions. He was observed being both productive and unproductive in his interactions in small groups. Even though his conceptual knowledge of the mathematics at hand appeared to be strong, sometimes his explanations of his work were difficult to understand.

During class session six, however, changes in his problem solving behaviors began to emerge. During group problem solving sessions, he began to articulate his conjectures and voice concern about establishing their validity. He was also observed monitoring the group problem solving sessions. These behaviors were particularly evident when the class was working on the Triangle Problem.

His resource base afforded him the ability to make sense of problems and construct viable solutions as evidenced by the fact that many of problem sets Ben turned in were free of errors. Study of his written assignments revealed that as the semester went on, he began to employ a greater variety of heuristics than were seen in the first phase of the semester or in the pre-instruction interview.
When working on the extended analyses tasks, Ben revealed that he could follow his curiosity and make powerful mathematical discoveries. This was especially evident from his work on the Evaporation Problem and the Box extensions.

By the last six class sessions, Ben appeared to have adopted many of the problem solving dispositions observed in more experienced problem solvers. In addition to monitoring behaviors, he was observed orienting himself to problems, conjecturing and planning solutions paths, executing those plans, and checking his solutions. Additionally, he began to voice an appreciation for seeing connections in and among various mathematical topics and ideas.

In some ways, Ben’s post-instruction measures were not very different than the pre-instruction measures. His performance on the pretest showed a modest 3.3% gain, but the results of his VAMS were very interesting. Recall that Ben had already been categorized as holding expert beliefs about mathematics according to the categories set down by Carlson, et al. (1998). His post instruction survey not only revealed a marked shift in his attitudes in the direction of a more expert view, but the survey items where these shifts occurred were the same on both Ben’s and Amy’s post-instruction VAMS.

In the post-instruction interview, Ben reported that his understanding of mathematics had been enriched by his experience in the class and that he experienced improvement in his problem solving abilities and felt even more confident in the topics that make up the high school mathematics curriculum. In addition, he credited the extended analyses tasks as being especially influential in helping him see the importance of connections and relationships between ideas and topics in mathematics. His final problem solving session corroborated the data from the last few classroom sessions,
namely that Ben had adopted many of the problem solving behaviors and dispositions observed in mathematicians when solving problems and when discussing problem solving.

This chapter has described the transformation of Ben’s mathematical knowledge and problem solving behaviors as he engaged in the instructional sequence described in Chapter Five of this paper. Numerous aspects of his problem solving behavior have been reported and described in the context of his written work on take-home assignments, group problem solving sessions, and extended analyses tasks. He is presented as a case study of a preservice mathematics teacher who was one of the highest performing students participating in the instruction. These data show that even though Ben had the benefit of a strong mathematical background, he was still able to make progress toward adopting the desired behaviors and dispositions outlined in the MPS Framework.
CHAPTER EIGHT
SUMMARY AND DISCUSSION

Summary

This study investigated the evolution of the problem solving behaviors and dispositions of two prospective secondary mathematics teachers (PSMTs). The investigation built on the problem solving research of Carlson and Bloom (2005) and used their Multidimensional Problem-Solving (MPS) Framework as the primary theoretical tool guiding research design, task construction and data analysis. The research questions that guided this investigation were:

1. What changes are evident in the problem solving behaviors of prospective high school mathematics teachers as they progress through an instructional sequence that engaged PSMTs in the practices of problem solving, mathematizing, generalizing, and extending?

2. What changes in other personal knowledge factors – confidence, views about mathematics, self efficacy, persistence etc. — are evident as PSMTs progress through an instructional sequence that engaged them in the practices of problem solving, mathematizing, generalizing, and extending?

3. In what ways did the instructional sequence influence changes in the problem solving behaviors and personal knowledge factors of PSMTs?

As a design experiment, this study described an instructional innovation that employed thought revealing activities and artifacts in conjunction with inquiry into theoretical considerations (Kelly & Lesh, 2002; Lesh, 2002), in this case regarding the
nature of the development of problem solving behaviors. In order to both describe and explain PSMT’s development, two case studies were provided: the Case of Amy is reported in Chapter Six, and the Case of Ben in Chapter Seven. The Multidimensional Problem-Solving (MPS) Framework was used to investigate and describe transformations in Amy and Ben’s problem solving behaviors and mathematical dispositions.

Summary of Amy’s Evolution

In Amy’s pre-instruction interview, she stated that she believed she was a good problem solver and was persistent in her efforts to solve difficult problems. She also expressed confidence in her abilities and knowledge of high school mathematics. Yet, her pre-instruction measures suggested that her conceptual understanding of high school mathematics (conceptions of rate, geometric reasoning and measurement in particular) were not well developed. VAMS results suggested that she held naïve views about mathematics, and her problem solving session revealed weaknesses in all areas of the MPS Framework. As reported in Chapter Six, these data suggest that Amy entered the experimental class with few of the tools needed to be successful when solving problems. In particular, it appeared that her conceptual understanding of mathematical certain concepts (e.g. calculus and ratios) was not well developed, and appeared to be based in rote procedures.

Amy’s work in the early phase of the semester was not surprising considering her performance on the pre-instruction measures. Her written work revealed that she did not persist in attempting to solve complex problems, and her statements during class showed that she could complete routine mathematical tasks but had difficulty justifying her actions mathematically.
However, starting with the sixth class session, changes in Amy’s mathematical behavior became apparent. Her assignments revealed improvement in her ability to construct productive solution paths and valid mathematical arguments. Amy began using heuristics other than guess-and-check. For example, she was observed examining cases and constructing diagrams. It appeared that Amy was becoming more persistent as the semester progressed, as evidenced by fewer unfinished homework problems. In her group interactions, she was observed persisting in making sense of her colleagues thinking and reflecting on the correctness of the group’s solution to particular tasks. As she was working through the Triangle problem, she expressed the realization that she needed to expend more effort in the orienting and planning phases of problem solving. Amy constructed isomorphic problems when completing the Extended Analysis Tasks, and showed excitement and curiosity in her efforts to extend these problems. Despite these improvements, Amy’s conceptual understanding of the mathematics needed to solve specific problems continued to present difficulties for her during her problem solving attempts. She appeared to be aware of her weak knowledge base; however her behaviors suggest that she was highly motivated to enrich her mathematical conceptions.

During the last segment of the instructional sequence, Amy’s problem solving abilities and mathematical dispositions continued to evolve. It appeared that she was better able to orient herself to problem situations and she used a greater variety of heuristics (e.g. constructing graphs, considering cases, searching for patterns) as she constructed her solution for a problem. She was also observed engaging in sense making, conjecturing and monitoring. An appreciation for abstract representations of situations and for alternative solution strategies emerged during this time period.
Amy’s post instruction measures revealed that although she made only modest gains on the content-focused exam, she exhibited a significant shift in her attitudes away from viewing mathematics as a body of knowledge about numbers and computations to a body of knowledge about ideas and relationships. Her pre-instruction VAMS characterized her as holding naïve views about mathematics, whereas the post-instruction VAMS results placed her in the expert category. It is interesting to note that Amy and Ben demonstrated shifts on the exact same VAMS items.

In her post-instruction interview, Amy reported that the course had been a positive influence on her understanding of mathematics and her problem solving abilities and she expressed greater confidence and persistence. She credited her work on the Extended Analysis Tasks as having influenced her ability to appreciate mathematical abstraction. She also stated that listening to the presentations of others’ mathematical explorations had helped her appreciate others’ ways of thinking. Data also supported that the experience of group problem solving and listening to others’ presentations had influenced her problem solving abilities. Changes were also evident in the area of monitoring as Amy routinely asked others to explain or justify their mathematical products. As the semester progressed her orienting behaviors improved and she was better able to articulate solution strategies. At times she was also observed monitoring her solution approach. In contrast to her pre-instruction interview, her post-instruction interview revealed that she had improved in her ability to reflect and hence, recognize errors in her thinking and computations.

However, there were areas in which there was little improvement in Amy’s problem solving abilities. Her post-instruction problem solving session revealed that her
poor conceptual understanding of certain mathematics topics continued to obstruct her efforts to perform mathematical tasks. Because of her weak understanding of relevant content, she was unable to make logical conjectures. She also failed access the mathematical resources that were needed for progressing in her solution problems.

Although quantitative measures did not reveal significant gains in conceptual understanding, the data support that aspects of Amy's problems solving behaviors and her thinking about mathematics in general had undergone an evolution.

**Summary of Ben’s Evolution**

In Ben’s pre-instruction interview, he stated that he believed that he was a good problem solver and was persistent in his efforts to solve difficult problems. He also expressed confidence in his knowledge of high school mathematics. The pre-instruction measures suggested that Ben was able to access his mathematical knowledge when confronting novel problems. His performance on the pretest and the pre-instruction problem solving session established that he also had a relatively strong understanding of conceptual and procedural knowledge of secondary mathematics, although his responses did reveal some weaknesses in his use of notation and understanding of the meaning of a variable. His views about the practice of mathematics and problem solving were aligned with those of the mathematical community. His performance on all three of the measures (pretest, VAMS, interview) revealed that he was one of the stronger students in the class.

Ben's work in the early phase of the semester was not surprising considering his performance on the pre-instruction measures; he was able to access key mathematical concepts when solving problems and appeared to be fairly confident and persistent in his efforts. However, analysis of data from the first five sessions revealed that Ben's work,
both individually and while interacting with peers sometimes revealed lapses in his engagement with the tasks. Sometimes he committed himself to a flawed plan of action with little reflection on the products or processes of his work; other times, he produced elegant solutions that relied on conceptual understanding of key ideas and regular monitoring of his processes and products.

During the second segment of the class changes in his problem solving behaviors began to emerge. While engaging in group problem solving sessions, he began to articulate his conjectures and voice concern about establishing their validity. He was also observed evaluating and assessing the group problem solving sessions. His resource base contributed to his effectiveness in making sense of novel problems. His persistence in constructing viable solutions was revealed in the numerous problem sets Ben turned in that were complete and free of errors. Study of his written assignments revealed that as the semester progressed, he began to employ a greater variety of heuristics than were seen in the first phase of the semester or in the pre-instruction interview. When working on the Extended Analyses Tasks, Ben stated that he enjoyed following his curiosity and making powerful mathematical discoveries. This was especially evident from his work on the Evaporation Problem and the Box Problem extensions. By the last six class sessions, Ben appeared to have adopted many of the problem solving dispositions observed in more experienced problem solvers. In addition to monitoring behaviors, he was observed regularly engaging in sense making as he attempted to orient himself to novel problems. He also became more and more effective in conjecturing and planning solutions paths, executing those plans, and checking his solutions. Additionally, he began to voice an
appreciation for seeing what he called connections in and among various mathematical
topics and ideas.

In some ways, Ben’s post-instruction measures were not very different than the
pre-instruction measures. His performance on the posttest showed a modest 3.3% gain,
however the results of his VAMS were very interesting. As previously reported, Ben’s
pre-instruction VAMS suggested that he had already been characterized as holding expert
beliefs about mathematics according to the categories set down by Carlson, et al. (1998).
His post instruction survey not only revealed a shift in his attitudes in the direction of a
more expert view, but the survey items where these shifts were revealed were the same
items on both Ben’s and Amy’s post-instruction VAMS.

In the post-instruction interview, Ben reported that his understanding of
mathematics had been enriched by his experience in the class, that he experienced
improvement in his problem solving abilities and felt even more confident in the topics
that make up the high school mathematics curriculum. In addition, he credited the
Extended Analyses Tasks as being especially influential in helping him see the
importance of connections and relationships between ideas and topics in mathematics.
His post-instruction problem solving session corroborated the data from the last few
classroom sessions, namely that Ben had adopted many of the problem solving behaviors
and dispositions observed in mathematicians when solving problems and when discussing
their mathematical work.

*Key Results*

Data analysis revealed six key results from the investigation of Amy’s and Ben’s
problem solving behaviors. These results are as follows:
1. Even though Amy and Ben entered the experimental class at differing ability levels, they both demonstrated improvements in their problem solving abilities.

2. The quality of Amy’s and Ben’s conceptions of mathematics impacted their problem solving abilities.

3. Extended Analysis Tasks revealed Amy’s and Ben’s mathematical curiosity and provided them with experience making mathematical discoveries.

4. The Triangle Problem appeared to be a pivotal experience for both Amy and Ben.

5. VAMS results revealed that Amy and Ben experienced specific shifts in their views about mathematics.

6. Amy and Ben reported that the class experience influenced how they thought about teaching mathematics.

In the following sections, I discuss each of these results and provide supporting evidence for my claims

Key finding #1

Even though Amy and Ben entered the experimental class at differing ability levels, they both demonstrated improvements in their problem solving abilities.

Data reported in Chapters Six and Seven provide evidence that both Amy’s and Ben’s problem solving behaviors improved as they progressed through the instructional sequence.
In Amy’s case, pre-instruction problem solving session revealed that she had difficulty making sense of novel problems, organizing the given information and articulating solution strategies. She showed little evidence of monitoring behavior and she appeared to be easily frustrated by failed attempts (see Chapter Six, Pre-instruction Interview, The problem solving session). Pre-instruction measures identified Amy as one of the weakest students in the class. As reported in Chapter Six, Amy’s problem solving behaviors steadily improved as she progressed through the instructional sequence. Despite the fact that Amy’s mathematical content knowledge often hindered her problem solving attempts, Amy demonstrated that she could effectively orient herself to the problem, set appropriate sub-goals, verbalize her solution path, and reflect on the reasonableness and validity of her constructions during the post-semester problem solving session (see Chapter Six, Post-instruction Interview, The problem solving session).

In contrast to Amy, pre-instruction measures identified Ben as one of the strongest students in the class. Nonetheless, his problem solving behavior also improved as he progressed through the problem solving sequence. Even though Ben’s pre-instruction problem solving session revealed that Ben was able to orient himself to the problem by accessing relevant mathematical concepts, map out and execute a plan, he experienced difficulties with his notational choices, and monitoring behaviors often failed to alert him to errors in this thinking or mathematical products (see Chapter Seven, Pre-instruction Interview, The problem solving session). Ben’s post instruction problem solving session revealed that Ben appeared to have adopted many of the problem solving dispositions observed in more experienced problem solvers. In particular, Ben’s
monitoring behaviors successfully alerted him to errors in reasoning, computations and poor notational choices, allowing him to adjust his solution strategies mid-path. In addition, he was observed regularly engaging in sense making as he attempted to orient himself to novel problems, making conjectures and planning solution paths, executing those plans, and checking his solutions (see Chapter Seven, Pre-instruction Interview, The problem solving session).

The instructional design outlined in Chapter Five appeared to be effective in providing the kinds of mathematical experiences needed to support their adoption of behaviors shown by Carlson and Bloom (2005) to be effective.

**Key finding #2**

The quality of Amy’s and Ben’s conceptions of mathematics impacted their problem solving abilities.

As reported in the findings particular to Amy, her mathematical conceptions often obstructed her ability to successfully solve problems. For example, Amy struggled to understand the proportional relationships defined in the Evaporation Problem (see Chapter Six, Transitions, Extended Analyses Tasks), and appeared to be unable to interpret the functions that were generated in the Box Problem (see Chapter Six, Final Phase, Extended Analysis Tasks). In the post-instruction problem solving session, she did not make use of the fact that the diagonals of a rectangle have the same length, which prevented her from completing the problem (see Chapter Six, Post-instruction Interview, The problem solving session). Therefore, despite the improvement in her problem solving abilities and shifts in her beliefs, her deficiencies in the areas of resources, as defined by Carlson and Bloom (2005), were often responsible for her failed solution attempts.
In contrast, it was apparent from the beginning that Ben’s mathematical conceptions supported his efforts to construct viable solution paths. In a majority of the problem solving activities he engaged in, Ben demonstrated that he could access relevant mathematical knowledge he could use to solve the problem. For example, when solving the Least Common Multiples Problem (Chapter Seven, pre-instruction interview) he accessed his understanding of factors and multiples to efficiently arrive at the solution. When solving the Composition Problem (Early in the semester, problem sets) his fluency with computational approaches allowed him to bypass some tedious algebraic manipulation. When extending the Box Problem (Final Phase, Extended Analysis Tasks), his mathematical understandings afforded him the ability to keep track of the variables and interpret the resultant graphs. Ben’s strengths in the area of resources, as defined by Carlson and Bloom (2005), helped him overcome some of the deficiencies in the problem solving process seen early in the instructional sequence, and supported his efforts throughout the semester.

While all the components making up the Multidimensional Problem-Solving Framework contribute to problem solving success, this finding suggests that the quality of the solver’s mathematical conceptions plays a critical role in the problem solving process. These data suggest also that rich mathematical conceptions can sometimes overcome deficiencies in other areas of the problem solving framework, the opposite is not true. That is, strengths in the others areas of the problems solving framework may not help the solver overcome impoverished resources. In other words, the results of the present study suggest that well developed conceptual knowledge is a necessary, but not sufficient condition for successful problem solving.
Carlson and Bloom (2005) reported that “Among the mathematicians we studied, well-connected conceptual knowledge appears to have influenced all phases of the problem solving process” (p. 70). The present study suggests this is true of undergraduate students as well.

*Key finding #3*

*Extended Analysis Tasks revealed Amy’s and Ben’s mathematical curiosity and provided them with experience making mathematical discoveries.*

Amy and Ben both credited the Extended Analysis Tasks with supporting the mathematical growth they experienced. Their curiosity also appeared to motivate their explorations and support their persistence as they worked towards solutions.

In Amy’s case, she used the Evaporation extension to explore a situation that she found puzzling. As mentioned in Chapter Six, Amy said, “I never understood how carpool lanes on the highway can cut traffic by 50%,” (see Chapter Six, Transitions, Extended Analyses Tasks) and that question appeared to guide some of her explorations. When Amy extended the Completing the Square extension, she seemed genuinely excited that she had derived the quadratic formula and the formulas for the vertex of a parabola (see Chapter Six, Transitions, Extended Analyses Tasks). It appeared that her extension of the problem was a genuine exploration that yielded (for her) unexpected results. When Amy presented her final project, she expressed excitement about patterns she noticed during her exploration (see Chapter Six, Final Phases, Final Project). For example, in her conclusions, she wrote:
One of the first things that we noticed was that 4 was our magic number. It appeared in patterns throughout our analysis. After playing with the numbers a little, we realized since we were mostly dealing with whole numbers, each side of the square had to be at least 1 unit long, therefore, an entire square consists of 4 units, which is why we see patterns of 4 throughout.

In the final project, Amy noticed the “magic number 4” and through her explorations was able to mathematically account for her discovery.

In most of her explorations, Amy tended to investigate extreme cases, but these explorations were valuable to her. In her end of the semester reflection, she stated, “If I could do something different, I would look at more extreme cases because I felt those gave a little more meaning to the problem.”

In Ben’s case, his curiosity was especially evident when he extended the Evaporation Problem (Chapter Seven, Transitions, Extended Analysis Tasks). Recall that during his exploration of the problem, he experimented with various solutions, noticing how the concentration of the substance varied with the amount of water that evaporated. As a result, he created a function that expressed the final concentration, and his graph of that function was used to explain the counter-intuitive nature of the solution of the original problem. Like Amy, Ben was excited when he found he had derived the quadratic formula (Chapter Seven, Transitions, Extended Analysis Tasks). In his extension, he wrote,

One thing that struck my curiosity relates to the quadratic formula, which is the discriminant and how the discriminant predicts, like, how many
solutions you are going to have. And it all seemed very magical to me until I did the extended analysis.

Ben reported that the Extended Analysis Tasks were especially beneficial for helping him discover mathematical relationships. When he reflected on his experiences with these tasks, he wrote, “Extended analysis problems are probably the most effective way of teaching mathematical relationships and concepts.”

As reported in Chapter Six and Chapter Seven, Ben’s explorations tended to be richer than Amy’s, primarily because his conceptual knowledge afforded him more avenues for exploration. Nonetheless, both Amy and Ben were able to follow their curiosity and make discoveries that were new to them. Therefore it appeared that the use of Extended Analysis Tasks in the instructional sequence provided both PSMT’s situations in which they were able to explore and discover mathematical relationships, and both reported that these experiences were very influential for them. In this way, Extended Analysis tasks supported not only problem solving, but mathematizing, generalizing and extending as well.

Key finding #4

The Triangle Problem appeared to be a pivotal experience for Amy and Ben.

The Triangle Problem appeared to be pivotal because the behaviors that emerged during the four class sessions allocated for solving the problem, presenting the solutions and reflecting on the experience, continued and improved as Amy and Ben progressed through the problem solving sequence.
As reported in Chapter Six, when Amy’s group began to work on the Triangle Problem (Transitions, Class work), she began to exhibit monitoring behaviors. As the other members of the group made conjectures and claims, Amy was observed asking members questions such as “Why did you want a perpendicular there?” and “What do you mean?” Amy’s questions had the effect of promoting sense making and reflection as the group grappled with the solution.

As reported in Chapter Seven, when the group began to work on the Triangle Problem (Transitions, Class work), Ben also began to exhibit monitoring behaviors. In Ben’s case, he regularly expressed that it was necessary for the group to prove their conjecture and he expressed concern that they may be heading off on a “crazy tangent.” These behaviors suggest that Ben was actively evaluating the effectiveness of the solution path under discussion.

When asked to reflect on the experience, Amy recognized that she needed to organize pertinent information and then use that information to solve the problem. In Ben’s reflection, he recognized that he had a tendency to “follow hunches” rather than reflect on the validity of the decisions that he made during his solution process. As the instructional sequence progressed, it appeared that Amy and Ben were able to use these insights to improve those problem solving behaviors.

As the semester progressed, Amy’s monitoring behavior took the form of asking group members to help her make sense of their conjectures, equations and constructions. Later, (as seen in the post-semester interview) she was able to ask herself these questions, which assisted her in her attempts to evaluate the products and progress as she solved problems. Ben’s monitoring behavior was generally in the form of critically evaluating
the validity of sub-goals and solution strategies. Later (as seen in the post-semester interview) his reflections alerted him to errors in his solution, allowing him to adjust his solution path.

It appeared that solving the Triangle Problem was effective in fostering the habit of monitoring. It appears that the subsequent act of reflecting on the experience revealed to the solver deficiencies in their problem solving behavior.

Key finding #5

VAMS results revealed that Amy and Ben experienced specific shifts in their views about mathematics.

Ben and Amy demonstrated the greatest shifts on VAMS items 4, 5, 10, 19, 20 and 24. The fact that their thinking shifted on the same six items suggests that the instructional sequence was especially influential in changing those beliefs about mathematics.

Items 4 and 5 pertain to teaching and learning mathematics. In item 4, each of their responses showed a shift away from thinking “My score on mathematics exams is a measure of how well I can do things the way they are done by the teacher or in some course materials”, and toward thinking “My score on mathematics exams is a measure of how well I understand the covered material”. In item 5, each of their responses showed a shift away from thinking “For me, doing well in mathematics courses depends on how well the teacher explains things in class,” and towards thinking “For me, doing well in mathematics courses depends on how much effort I put into studying”. These shifts are interesting because these issues were not part of the curriculum and were not addressed in class, suggesting that the instructional innovation had the effect of instilling the value of
conceptual understanding of mathematics over following procedures, and shifting some of the responsibility for student learning from the teacher to the student.

Items 10, 19 and 24 pertain to the nature of mathematics as a body of knowledge. For item 10, each of their responses revealed a shift in thinking away from “Mathematical formulas provide ways to get answers” and towards thinking that “Mathematical formulas express meaningful relationships among variables.” For item 19, each of their responses revealed a shift in thinking away from “Collecting and graphing real world data is useful for obtaining numerical answers to specific problems” and towards thinking that “Collecting and graphing real world data is useful for determining patterns and making general predictions.” Each of their responses on Item 24 revealed shifts in their thinking away from “In order to prove a mathematical theorem one must produce evidence from the physical world,” towards thinking that “In order to prove a mathematical theorem one must provide a logically sound argument”.

Item 20 pertained to the problem solving process. Amy and Ben showed shifts away from thinking that “For me, making unsuccessful attempts when solving a mathematics problem is an indication of my incompetence in mathematics,” towards thinking that “For me, making unsuccessful attempts when solving a mathematics problem is a natural part of my pursuit of a solution to the problem.” These shifts suggest that through their participation in the instructional sequence, they both came to accept errant solution paths a part of the problem solving process.

These shifts provide evidence that the instructional design influenced the mathematical beliefs of Amy and Ben.
Key finding #6

Amy and Ben reported that the experience influenced how they thought about teaching mathematics.

As prospective secondary mathematics teachers (PSMTs), Amy and Ben occasionally made mention of tutoring and field experiences they participated in, as well as their future as high school mathematics teachers.

In her written reflections submitted at the end of the instructional sequence, Amy made several references to her future as a mathematics teacher. She wrote:

I thought this class was very beneficial for us because we went over mathematical topics that are challenging and those problems that have common mistakes made in the high school classroom. . . . When I’m a teacher I will present different ways that you can approach the same answer and also let students work together in groups to solve problems. . . . Overall, I found this class to be helpful in preparing me to become a teacher.

Amy also stated, during the post instruction interview, “as I sit down with the students, I definitely, I try to show them [multiple solutions].” These statements suggest that the instructional sequence changed some of the ways Amy conceived of teaching mathematics. In particular, she revealed that multiple solution paths and productive group work might be instructive for her future students as they were for her.

In Ben’s written reflections submitted at the end of the instructional sequence, he also made several references to his future as a teacher.
I thought . . . everything we did had a purpose and was usually to help in self-discovery about mathematics, teaching mathematics and ourselves. I learned a lot about how effective it is to show connections in math. … I learned a lot about my own strategies and my own thresholds for frustration. This will help me relate to my students and understand where they are coming from. I think I have gained a lot of confidence in my own math skills and that will spill over while I teach.

Ben, in discussing a lesson he was preparing for his field experience, said:

I would like to do just connections the whole time, and like do examples, but, towards the end of the lecture its more, like examples. But if I had more time it would just be nice to be able to expose everything behind it so that they make all the connections and understand it better so that when they’re making mistakes they know where it’s coming from

These statements suggest that the instructional sequence changed some of the ways Ben conceived of teaching mathematics. Ben’s statements suggest he found that the discovery of what he called “connections” was valuable to him and he wished to expose his students to these connections as well.

Comparison of pre and post VAMS provides supporting evidence that there have been shifts in the ways Amy and Ben view the teaching and learning of mathematics. Recall that VAMS items 4 and 5 pertained to teaching and learning mathematics. In item 4, their responses revealed a shift away from thinking “My score on mathematics exams is a measure of how well I can do things the way they are done by the teacher or in some course materials”, and toward thinking “My score on mathematics exams is a measure of
how well I understand the covered material”. In item 5, their responses revealed a shift away from thinking “For me, doing well in mathematics courses depends on how well the teacher explains things in class,” and towards thinking “For me, doing well in mathematics courses depends on how much effort I put into studying”.

The data suggested that aspects of the instructional design, such as eliciting multiple solutions for the same problems, effective group work, may have had the effect of changing the PSMT’s conceptions of teaching mathematics.

Summary of findings

The analyses of data provided in Chapters Six and Seven support the following findings. Despite differing ability levels, both Amy and Ben demonstrated problem solving behavior that was more aligned with that of mathematicians as described in the Carlson and Bloom (2005) study. Amy’s and Ben’s individual evolutions suggest that conceptual knowledge is a necessary but not sufficient condition for problem solving success. Extended Analysis Tasks appeared to foster curiosity and support resulting explorations. Engagement in the Triangle Problem activity appeared to be a pivotal aspect of the instructional sequence since changes in Amy’s and Ben’s problem solving behaviors emerged in conjunction with this activity. Comparison of pre and post-instruction VAMS provides evidence of shifts in these PSMT’s beliefs about mathematics. This is encouraging since Thompson (1992) found that teachers’ beliefs can be quite resistant to change. Finally, although the teaching and learning of mathematics was not formally addressed in the instructional sequence, Amy and Ben felt their experiences influenced how they viewed teaching.
The Research Questions Addressed

In this section, I use the results of the present study to answer the research questions that guided this investigation.

Research Question #1

What changes are evident in the problem solving behaviors of prospective high school mathematics teachers as they progress through an instructional sequence that engaged PSMTs in the practices of problem solving, mathematizing, generalizing, and extending?

The data showed that both Amy and Ben experienced similar evolutions in their problem solving behaviors as they progressed through the instructional sequence. In terms of the Multidimensional Problem Solving Framework, data analysis revealed that their problem solving behaviors improved in the following areas:

- Orienting behavior
- Planning behavior
- Checking behavior
- Monitoring behavior
- Resources
- Heuristics

A discussion of the changes exhibited in each of these areas is provided in the following section. Changes in the affective domain of problem solving were also evident, and are discussed separately in Research Question #2.

Orienting behavior
Amy’s pre-instruction problem solving session revealed that she experienced difficulty accessing appropriate relevant mathematical knowledge needed when confronting unfamiliar problems. As mentioned in Chapter Six, this was especially evident as she attempted to make sense of the Square Paper Problem. When she reflected on her experience solving the Triangle Problem, she admitted that this was an area where she needed to improve. In the post-instruction problem solving session Amy demonstrated improvement in the phase of orienting by taking more time and effort in making sense of the problem, organizing the given information and making logical constructions as a result.

Ben’s strong conceptual knowledge base afforded him more success in the orienting phase of the problem solving cycle, yet data from the first segment of the instructional sequence revealed instances when his lapses in the orienting phase resulted in inefficient or unproductive solution paths. As the instructional sequence progressed, however, it appeared that Ben routinely oriented himself to problem statements by engaging in sense-making, accessing relevant mathematical knowledge and producing logical constructions

Planning behavior

Analysis of Amy’s pre-instruction problem solving session produced little evidence that Amy made conjectures, articulated solution strategies or made informed decisions regarding the validity of effectiveness of any particular solution path. As she progressed in the instructional sequence, these behaviors emerged (although the quality of her content knowledge routinely limited their effectiveness). Analysis of Amy’s post-
instruction problem solving session provided evidence that Amy made conjectures and verbalized solution strategies.

Ben’s pre-instruction problem solving session did produce evidence that Ben could produce a solution strategy, but there was little evidence that he made and tested conjectures or that he made informed decisions regarding the validity or effectiveness of his solution path. When Ben and his group engaged in the Triangle Problem, he began to clearly articulate his conjectures and voice concern about establishing their validity. In his written reflection of the experience, he admitted that he tended to “follow hunches” and needed to expend more effort reflecting on the effectiveness or validity of his decisions. The final segment of the instructional sequence and the post-instruction interview provided evidence that Ben engaged in the conjecture-imagine-evaluate loop and considered various solution approaches in the process of seeking solutions to problems.

Checking behavior

Amy’s pre-instruction problem solving session revealed that she depended on the authority of others (the interviewer, the instructor or other group members) to provide validation of her mathematical constructions. It appeared that she steadily became able to check the validity or reasonableness of her solutions as evidenced by a continual improvement on her scores on submitted assignments. The post-instruction problem solving session provided evidence that Amy engaged in checking behavior and relied on the assessment of her constructions to determine the validity of her work products.

Monitoring behavior
Amy’s pre-instruction problem solving session provided very little evidence that she monitored her thinking, her progress or efficacy of her solutions. When Amy and her group engaged in the Triangle Problem, however, Amy was observed asking monitoring-type questions. Questions such as “what does that mean?” and “why would you do that?” had the effect of prompting others to reflect on their products and progress. During the post-instruction interview, Amy was observed asking herself these types of questions. Additionally, reflections on her solution path alerted her to an error in her thinking when solving the Bottle Problem.

Even though Ben’s pre-instruction problem solving session revealed some evidence that he reflected on his thinking and on the validity of his solutions, the monitoring he engaged in failed to alert him to poor notational choices, errors in reasoning and errors in computations, suggesting limitations in his ability to engage in on-line monitoring and sense-making. As he progressed in the instructional sequence, Ben was observed critically evaluating the progress and products of the group’s efforts to solve difficult problems. The post-instruction problem solving session revealed that Ben’s monitoring behaviors successfully alerted him to problems with the variables he chose in the Smallest Area Problem.

Resources

Amy’s pre-instruction measures and her problem solving behaviors early in the instructional sequence revealed that she often experienced difficulty determining what mathematical facts and concepts she needed to confront the problem she was working. Even though the quality of Amy’s mathematical conceptions could impede her problem solving efforts, review of her submitted problem sets and her discussions during group
work suggested that she was more successful accessing relevant mathematical information as the instructional sequence progressed.

From the beginning of the experiment, it appeared that Ben’s strong conceptual knowledge base supported his efforts to access appropriate mathematical facts and concepts. There was evidence, however, that he experienced some difficulty accessing his knowledge of calculus when it was called for. As he progressed in the instructional sequence and participated in the post-instruction problem solving session, it appeared that Ben was routinely successful accessing the mathematics needed to solve the problems. Ben stated that he found the instructional sequence helped him “make connections” between mathematical topics and ideas, and these connections may have contributed to his success in this area.

Heuristics

Early in the intervention, Amy’s primary heuristics were guess-and-check and constructing equations. As the semester progressed, she demonstrated using the following heuristics: making sketches and diagrams, examining cases, setting sub-goals and constructing graphs.

Early in the intervention, Ben’s primary heuristics were sketching graphs and constructing equations. As the semester progressed he made use of an increased number of heuristics that included the following: making sketches and diagrams, examining cases, setting sub-goals, employing specific algorithms, creating computer programs and constructing graphs.

In summary, changes in Amy’s and Ben’s problem solving behavior during the orienting, planning and checking phases of the problem solving cycle became more
aligned with those observed in mathematicians in the Carlson and Bloom (2005) study. In addition, they appeared to be better able to make use of heuristics, access relevant resources and engage in monitoring behavior.

Research Question #2

*What changes in other personal knowledge factors – confidence, views about mathematics, self efficacy, persistence etc., — are evident as PSMTs progress through an instructional sequence that engaged them in the practices of problem solving, mathematizing, generalizing, and extending?*

Analysis of the data presented in the present study suggests that Amy and Ben experienced changes in the following personal knowledge factors:

- Views about mathematics.
- Persistence
- Integrity
- Curiosity

A discussion of the changes exhibited in each of these areas is provided in the following section.

Views about mathematics.

A detailed discussion of the shifts Amy and Ben demonstrated on VAMS is discussed in detail in the Key Results section of this chapter, under Key Finding #5. In general, comparison of their pre and post-instruction VAMS results revealed that after the instructional sequence they were more likely than before the instruction to view mathematics as knowledge of ideas and relationships, and that teaching and learning mathematics entails understanding mathematical concepts as opposed to merely
performing calculations. Comparison of pre and post-instruction VAMS results also suggests that Amy and Ben came to value conceptual understanding of mathematics over following procedures, and shifting some of the responsibility for student learning from the teacher to the student.

Persistence

In the pre-instruction interview, both Amy and Ben stated that they felt they were persistent in their efforts to solve difficult problems. It is interesting to note that during the pre-instruction problem solving session, neither one was observed being persistent. They expressed different strategies to employ when they got stuck on a problem, however, Amy tended to look for a formula to help her out, whereas Ben reported that he was likely to review his class notes and examples in an effort to gain understanding. Amy’s difficulty persisting in her problem solving effort was evidenced by the many incomplete assignments she submitted in the first segment of the instructional sequence. Amy’s improved persistence was revealed by submitting complete problem sets. During the post-instruction problem solving session, she provided evidence of persistence by revisiting problems she had previously abandoned. Ben’s improved persistence was evidenced by the quality of the work on his Extended Analysis Tasks. During the post-instruction problem solving session, he also provided evidence of persistence by revisiting and solving problems he had previously abandoned.

Integrity

As previously mentioned in Chapter Two, Debellis (1998) refers to mathematical integrity as the solver’s sense of whether the solution is correct or justified and a
willingness to admit mathematical shortcomings. Carlson and Bloom (2005) identified this as one of the traits observed in mathematicians.

As an example of Ben’s integrity, recall that when Ben’s group was confronted with the Triangle Problem (Transitions, Class work) members of the group decided that they needed to find the centroid of the triangle in order to solve the problem. Ben, however, continually stated that they would need to prove that the centroid was the point they needed to solve the problem. From this point on, Ben was observed periodically reflecting on the validity of his conjectures while solving problem.

Amy also demonstrated mathematical integrity. As discussed in Chapter Six, she was observed occasionally questioning claims made during group problem solving sessions.

Curiosity

As reported under Key Finding #3, engagement in Extended Analysis Tasks provided evidence that Amy and Ben showed gains in their level of curiosity about the mathematics at hand. In addition, both were observed using the tasks to follow their curiosity and make new (for them) mathematical connections and discoveries.

In summary, the changes that were evident in personal knowledge factors as Amy and Ben progressed through the instructional sequence included confidence, persistence, integrity and curiosity. In addition, they demonstrated shifts in their views about mathematics.

Research Question #3

In what ways did the instructional sequence influence changes in the problem solving behaviors and personal knowledge factors of PSMTs?
The instructional design, as explained in Chapter Five, provided the environment that fostered changes in the problem solving behaviors and personal knowledge factors of PSMTs. In the following, I describe in what ways the instructional sequence influenced the changes reported in the first two research questions.

The instructional sequence provided many problems for the PSMTs to solve. Some were solved individually, in the form of assigned problem sets. Some were solved in group problem solving sessions. Based on the research data already presented, both activities appeared to support improved problem solving behavior. Assigned problems provided opportunities for Amy and Ben to scan their knowledge base to access relevant mathematics and to practice the strategies they were exposed to in class. As Ben revealed during his post-instruction interview, “When you give us problems in class. Like we don’t know – usually you get a problem, it’s in a section and you know I’m going to be using what I just learned. So it’s nice to have all these problems in our class.” Ben also stated, “I got a lot of practice and gained a lot of confidence that will allow me to teach better. This confidence was boosted further through constant exposure to frustrating math problems.”

Student’s presentations of the various problems appeared to be especially beneficial for Amy. As mentioned before, Amy revealed that the presentations helped her because “the different approaches” provided her with solution strategies “that I wouldn’t typically think of”.

Group problem solving sessions appeared to provide them with insights into others’ thinking as well as models for alternative solution paths. Amy said “I liked [group work] because I would do one thing, someone would do something else. And like, “So
how did you do this?” and try and follow their work and they would try and explain it to me.” Ben’s comments were similar to Amy’s:

[S]ometimes one person would just have a block, like when we talk about breaking up your problem solving skills into those four levels? Some person might not have, like, the tools that you need, but as soon as you remind them of that certain tool, they’re like – oh yeah, oh okay. So it’s nice to have other people even if you just need to remind them quickly about it, it’s nice so you don’t get stuck all the time. And the ideas come out quicker, so…it’s nice.

Group problem solving sessions also appeared to support the development of greater persistence. The nature of the instructional sequence was such that everyone in the class had time to complete one activity before moving on to the next. No one could “give up” because the instructor did not move on to the next task until all groups had sufficient time to find a solution. It appeared that solving the Triangle Problem (which spanned four class sessions) fostered the habit of monitoring; and the subsequent act of reflecting on the experience appeared to reveal to the solver deficiencies in their problem solving behavior.

Finally, as reported in Key finding #3, Extended Analysis Tasks also contributed to the changes experienced by Amy and Ben. For Amy, the tasks seemed to provide her with opportunities to use mathematics for something more than to find numerical solutions to problems. The following excerpt is from reflection on her final project.
After all our extended analyses as well as our final project, I liked how we were able to look at the problems from different aspects rather than the "normal/traditional" way and most of the times we would conclude something interesting about it. Mathematically I really learned that graphs could give you a lot of different information, so I now like to set up different functions so I can see how they affect the problem. Also, I thought I did well parameterizing the problem and I liked parameterizing the problem because it allowed me to set up a formula for any given value and to see what would happen with a different value. If I could do something different, I would look at more extreme cases because I felt those gave me a little more meaning to the problem.

For Ben, Extended Analysis Tasks were especially beneficial for helping him discover mathematical relationships. When he reflected on his experiences with these tasks, he wrote, “Extended analysis problems are probably the most effective way of teaching mathematical relationships and concepts.” Amy and Ben were able to follow their curiosity and make discoveries that were new to them. Therefore it appeared that the use of Extended Analysis Tasks in the instructional sequence provided both PSMTs situations in which they were able to explore and discover mathematical relationships, and both reported that these experiences were very influential for them. In this way, Extended Analysis tasks supported not only problem solving, but mathematizing, generalizing and extending as well.
Discussion

Limitations of the Study

This section addresses the limitations of this study. The first part provides a discussion of the use of the NAEP items; the second section discusses issues of the instruction during the design experiment and the last section addresses the limitations inherent in using case studies.

The use of NAEP items for the pretest and posttest.

As reported in Chapter Five, one of the over-arching goals of the instructional sequence was to promote deep and well developed understandings of mathematics. Although the use of the NAEP items appeared to be appropriate for establishing Amy’s and Ben’s proficiencies in the mathematics assessed by the instrument, it did not provide much insight into the quality of their mathematical conceptions. Therefore, it was not a useful tool for making claims about knowledge growth as a result of the instructional innovation. In this way, my selection of this particular research instrument limited my investigation.

The quality of the instruction

The research design and the framework did not support close study of the interactions and interventions of the instructor or on her impact on Amy’s and Ben’s transformations.

In the period of time between the collection of data and the construction of this dissertation, I have had the opportunity to observe many other mathematics teachers at work in their classrooms. This experience, combined with the careful review of my own teaching videos as I analyzed the data for the present study, has led me to believe that my
instruction was an important component of the evolution of the subjects. Unfortunately, this component is only briefly discussed in Chapter Five and was never built into the research design.

Case study methodology

The findings reported here would be much more compelling had I described the evolutions of the problem solving behaviors for all 19 students enrolled in the experimental class. In this way, using case studies reduced the scope of my claims from the class as a whole, to applying to two students in the class. However, had I pursued the task of studying the class as a whole, I would not have been able to provide the fine grained analyses provided in Chapters Six and Seven.

Implications and Recommendations

In this section, I provide a discussion of the implications of this study in the areas of problem solving instruction, mathematics teachers’ beliefs and mathematics teacher preparation. My discussions include recommendations for future avenues of investigation.

Problem solving instruction

As reported in Chapter Two, The National Council of Teachers of Mathematics (NCTM) (2000) has identified mathematical problem solving as being central to the study of mathematics.

Problem solving is an integral part of all mathematics learning. In everyday life and in the workplace, being able to solve problems can lead to great advantages. However, solving problems is not only a goal of learning mathematics but also a major means of doing so. Problem solving
should not be an isolated part of the curriculum but should involve all Content Standards. (p. 52)

The findings of the present study provide guidance for problem solving instruction. Recall that Amy and Ben reported that the following aspects of the instructional design influenced their transformations as problem solvers.

- A lot of practice solving many problems.
- The chance to see multiple solution paths for any given problem.
- Productive group work.
- Extended Analysis Tasks.

The data suggest that these practices, implemented in any classroom, could support the development of high quality problem solving behavior.

It is important to note that the findings of the present study also suggest that attention must be paid to developing conceptual understanding of mathematics as well. As reported in Key finding #2, it appears that improved problem solving behaviors cannot overcome weak or fragmented content knowledge when confronting difficult or novel problems.

Future research would be needed to determine in what ways the mathematical activities and instructional practices listed above should be implemented in mathematics classes.

*Mathematics teachers’ beliefs*

It appears that aspects of the instructional design, such as eliciting multiple solutions for the same problems, effective group work, and making connections between mathematical ideas and mathematical topics had the effect of changing the PSMTs’
conceptions of teaching mathematics. Thompson (1992) has suggested that teachers’ conceptions of teaching mathematics can be resistant to change. Research would need to be conducted on PSMTs engaging in an instructional sequence such as the one described in this dissertation to determine if the changes in beliefs reported here were evident in their teaching practice.

The mathematical education of teachers

As reported earlier, even though Amy demonstrated improved problem solving behavior and encouraging shifts in her views about mathematics, she was unable to completely overcome some deficiencies in her mathematical conceptions. Apparently Amy’s success in the mathematics courses required for a teaching certificate did not support the construction of strong, well developed conceptions of the mathematical topics she would be teaching.
REFERENCES


APPENDIX A

NAEP ITEMS AND DOCUMENTATION
This first section of this appendix provides the pretest and posttest used in this investigation. This is followed by NAEP Mathematical Framework definitions provided by NCES for mathematics content areas and ability types

The pretest and posttest

The following is the pretest and posttest administered to the subjects in this study.

1. A circle with diameter 10 centimeters is to be cut from a square of paper 10 centimeters on a side. Of the following, which is closest to the amount of paper left over after the circle is cut out?

   A) 9 square centimeters
   B) 21 square centimeters
   C) 24 square centimeters
   D) 69 square centimeters
   E) 84 square centimeters

2. Suppose that \( a_1, a_2, a_3, \ldots \) is the sequence of numbers such that \( a_1 = 3, a_2 = \sqrt{a_1} + 1, a_3 = \sqrt{a_2} + 1, \) and, in general, \( a_{n+1} = \sqrt{a_n} + 1 \) for all \( n \geq 1 \). To the nearest hundredth, the value of \( a_5 \) is

   A) 1.63
   B) 2.62
   C) 2.73
   D) 3.24
   E) 5.73

3. \( \cos^2(3x) + \sin^2(3x) = \)

   A) 0
   B) 1
   C) 3
   D) 6
   E) 9
4. If triangles $ADE$ and $ABC$ shown in the figure above are similar, what is the value of $x$?

A) 4  
B) 5  
C) 6  
D) 8  
E) 10

5. The population of the United States is approximately 250 million, and the national debt is approximately 4 trillion dollars. If this debt were divided equally among the population, what would be the debt, in dollars, per person?

A) 16  
B) 1,600  
C) 16,000  
D) 1,600,000  
E) 16,000,000

6. A rectangular pool 24 feet long, 8 feet wide, and 4 feet deep is filled with water. Water is leaking from the pool at the rate of 0.4 cubic foot per minute. At this rate, how many hours will it take for the water level to drop 1 foot?

A) 4  
B) 8  
C) 12  
D) 16  
E) 32

7. It takes 28 minutes for a certain bacteria population to double. If there are 5,241,763 bacteria in this population at 1:00 p.m., which of the following is closest to the number of bacteria in millions at 2:30 p.m. on the same day?

A) 80  
B) 40  
C) 20  
D) 15  
E) 10
8. In \( \triangle ABC \) shown above, \( AC = 12 \). What is the length of segment \( BD \)?

A) \( 3\sqrt{2} \)

B) \( 3\sqrt{3} \)

C) 6

D) \( 6\sqrt{2} \)

E) \( 6\sqrt{3} \)

9. For what value of \( x \) is \( 8^x = 16^x \)?

A) 3

B) 4

C) 8

D) 9

E) 12

10. What is the distance between the points \((2, 10)\) and \((-4, 2)\) in the \(xy\)-plane?

A) 6

B) 8

C) 10

D) 14

E) 18

11. Suppose \( 4r = 3s = 10t \), where \( r, s, \) and \( t \) are positive integers. What is the sum of the least values of \( r, s, \) and \( t \) for which this equality is true?

A) 7

B) 17

C) 41

D) 82

E) 120
12. In the xy-plane, a line parallel to the x-axis intersects the y-axis at the point (0, 4). This line also intersects a circle in two points. The circle has a radius of 5 and its center is at the origin. What are the coordinates of the two points of intersection?

A) (1, 2) and (2, 1)
B) (2, 1) and (2, -1)
C) (3, 4) and (3, -4)
D) (3, 4) and (-3, 4)
E) (5, 0) and (-5, 0)

13. The entire circle shown above represents a total of 2,675 radios sold. Of the following, which is the best approximation of the number of radios represented by the shaded sector of the circle?

A) 70
B) 275
C) 985
D) 25,880
E) 98,420

14. In the graph above, each dot shows the number of sit-ups and the corresponding age for one of 13 people. According to this graph, what is the median number of sit-ups for these 13 people?

A) 15
B) 20
C) 45
D) 50
E) 55
15. In a group of 1,200 adults, there are 300 vegetarians. What is the ratio of nonvegetarians to vegetarians in the group?

A) 1 to 3  
B) 1 to 4  
C) 3 to 1  
D) 4 to 1  
E) 4 to 3

16. A certain company keeps a list of 50 employees and their annual salaries. When the salary of the very highly paid president is added to this list, which of the following statistics is most likely to be approximately the same or nearly the same for the original list and the new list?

A) The highest salary  
B) The range  
C) The mean  
D) The median  
E) The standard deviation

17. The figure above shows the graph of \( y = f(x) \). Which of the following could be the graph of \( y = |f(x)| \)?

A) 

B) 

C)
18. In the figures above, the radius and height of each right circular cylinder are given. If \( w \), \( x \), and \( y \) represent the respective volumes of the cylinders, which of the following statements is true?

A) \( y = w = x \)
B) \( y \leq x \leq w \)
C) \( y \leq w \leq x \)
D) \( w \leq y \leq x \)
E) \( w \leq x \leq y \)
19. In right triangle $ABC$ above, $\cos A =$

A) $3/5$
B) $\frac{3}{4}$
C) $4/5$
D) $4/3$
E) $5/3$

20. The least common multiple of 8, 12, and a third number is 120. Which of the following could be the third number?

A) 15
B) 16
C) 24
D) 32
E) 48

21. The length of a rectangle is 3 more than its width. If $L$ represents the length, what is an expression for the width?

A) $3 \div L$
B) $L \div 3$
C) $L \times 3$
D) $L + 3$
E) $L - 3$

The following question refers to the graph shown below.
22. What is the value of \( f(g(1)) \)?

A) 2  
B) 4  
C) 5  
D) 6  
E) 8

23. The figure above shows the display on a scientific calculator. The value of the displayed number is between which of the following pairs of numbers?

A) 0.04 and 0.05  
B) 0.4 and 0.5  
C) 4.0 and 5.0  
D) 40.0 and 50.0  
E) 400.0 and 500.0

24. A contractor is building 5 different model homes on 5 adjacent lots on one side of a street. If 1 house is to be built on each lot, how many different arrangements of the 5 houses are possible?

A) 120  
B) 60  
C) 25  
D) 10  
E) 5

25. In the figure above, points \( A, E, \) and \( H \) are on a plane that intersects a right prism. What is the intersection of the plane with the right prism?

A) A line  
B) A triangle  
C) A quadrilateral  
D) A pentagon  
E) A hexagon
26. The postal rate is 25 cents for the first ounce and 20 cents for each additional ounce or part of an ounce. What would it cost to mail a package that weighs 6.8 ounces?

A) $1.25  
B) $1.40  
C) $1.45  
D) $1.70  
E) $1.75

27. Semicircles are constructed on the sides of an equilateral triangle, as shown in the figure above. Of the following, which best approximates the sum of the lengths of the three darkened arcs?

A) 4.4258  
B) 4.7124  
C) 6.0000  
D) 6.7124  
E) 9.4258

28. A savings account earns 1 percent interest per month on the sum of the initial amount deposited plus any accumulated interest. If a savings account is opened with an initial deposit of $1,000 and no other deposits or withdrawals are made, what will be the amount in this account at the end of 6 months?

A) $1,060.00  
B) $1,061.52  
C) $1,072.14  
D) $1,600.00  
E) $6,000.00
29. The graphs of \( y = f(x) \) and \( y = g(x) \) for \( 0 \leq x \leq 10 \) are shown in the figure above. For how many values of \( x \) is the product \( f(x) g(x) = 0 \) for \( 0 \leq x \leq 10 \)?

A) Two  
B) Four  
C) Five  
D) Six  
E) Seven

30. A baseball card increases in value according to the function, \( b(t) = \frac{5}{2} t + 100 \), where \( b \) gives the value of the card in dollars and \( t \) is the time (in years) since the card was purchased. Which of the following describe what \( \frac{5}{2} \) conveys about the situation?

I. The card's value increases by $5 every two years.  
II. Every year the card's value is 2.5 times greater than the previous year.  
III. The card's value increases by \( \frac{5}{2} \) dollars every year.

a) I only  
b) II only  
c) III only  
d) I and III only  
e) I, II and III

NAEP Mathematical Framework

The 1990–2003 NAEP Mathematics Framework was used to develop the 1990, 1992, 1996, 2000, and 2003 assessments. Like all NAEP assessment frameworks, it was developed by the National Assessment Governing Board (NAGB). The 1990–2003 NAEP mathematics framework was influenced by the National Council of Teachers of Mathematics (NCTM) Curriculum and Evaluation Standards for School Mathematics.

Mathematical Content Strands

Definitions for the mathematical content strands may be found at the following website: http://www.nces.ed.gov/nationsreportcard/mathematics/contentstrands.asp
Mathematical Abilities

Definitions for the three content abilities may be found at the following website:


The 1990–2003 framework described five broad strands of mathematics content, as follows:

i). Number Sense, Properties, and Operations;

ii) Measurement;

iii.) Geometry and Spatial Sense;

iv.) Data Analysis, Statistics, and Probability; And

v.) Algebra and Functions.

Figure 141: NAEP Mathematical Framework
Analysis of pre and posttest

All students enrolled in the class were administered the pretest and the posttest. In this section, results of the pretest and posttest are provided, as well as some analysis of the scores.

When comparing the pretest and posttest scores (Table 24), students enrolled in the experimental class demonstrated statistically significant gains on the posttest with a $p$ value of less than .001. The average gain was 5.9% and the normalized gain was 7.6%. This shows that not only did the students come into the class knowledgeable in the content area of high school mathematics, but they were also able to make substantial gains on this instrument in the course of the semester.

Table 25 contains the pre-posttest items that evidenced the greatest gains.
Table 24

*Pretest and Posttest Scores of Students*

<table>
<thead>
<tr>
<th>Student</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amy</td>
<td>22</td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
<td>21</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>23</td>
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<td>4</td>
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<td>6</td>
<td>27</td>
<td>29</td>
</tr>
<tr>
<td>7</td>
<td>27</td>
<td>28</td>
</tr>
<tr>
<td>8</td>
<td>28</td>
<td>28</td>
</tr>
<tr>
<td>9</td>
<td>23</td>
<td>26</td>
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<tr>
<td>10</td>
<td>13</td>
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</tr>
<tr>
<td>11</td>
<td>18</td>
<td>21</td>
</tr>
<tr>
<td>12</td>
<td>27</td>
<td>28</td>
</tr>
<tr>
<td>Ben</td>
<td>27</td>
<td>28</td>
</tr>
<tr>
<td>14</td>
<td>21</td>
<td>23</td>
</tr>
<tr>
<td>15</td>
<td>24</td>
<td>26</td>
</tr>
<tr>
<td>16</td>
<td>24</td>
<td>28</td>
</tr>
<tr>
<td>17</td>
<td>27</td>
<td>28</td>
</tr>
<tr>
<td>Ave</td>
<td>23</td>
<td>25</td>
</tr>
</tbody>
</table>
Table 25

*Items Showing Gains > 15%*

<table>
<thead>
<tr>
<th>Item</th>
<th>Question</th>
<th>Percent change</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Suppose that $a_1, a_2, a_3, \ldots$ is the sequence of numbers such that $a_1 = 3, a_2 = \sqrt{a_1} + 1, a_3 = \sqrt{a_2} + 1$, and, in general, $a_{n+1} = \sqrt{a_n} + 1$ for all $n \geq 1$. To the nearest hundredth, the value of $a_5$ is</td>
<td>17.6%</td>
</tr>
<tr>
<td>3</td>
<td>$\cos^2(3x) + \sin^2(3x) =$</td>
<td>17.6%</td>
</tr>
<tr>
<td>6</td>
<td>A rectangular pool 24 feet long, 8 feet wide, and 4 feet deep is filled with water. Water is leaking from the pool at the rate of 0.4 cubic foot per minute. At this rate, how many hours will it take for the water level to drop 1 foot?</td>
<td>17.6%</td>
</tr>
<tr>
<td>15</td>
<td>In a group of 1,200 adults, there are 300 vegetarians. What is the ratio of nonvegetarians to vegetarians in the group?</td>
<td>17.6%</td>
</tr>
</tbody>
</table>
16  A certain company keeps a list of 50 employees and their annual salaries. When the salary of the very highly paid president is added to this list, which of the following statistics is most likely to be approximately the same or nearly the same for the original list and the new list?

18  In the figures above, the radius and height of each right circular cylinder are given. If \( w, x, \) and \( y \) represent the respective volumes of the cylinders, which of the following statements is true?

24  A contractor is building 5 different model homes on 5 adjacent lots on one side of a street. If 1 house is to be built on each lot, how many different arrangements of the 5 houses are possible?
APPENDIX B

VAMS ITEMS AND DOCUMENTATION
This first section of this appendix provides the Views About Mathematics Survey (VAMS) used in this investigation. This is followed by the analysis schema provided by Carlson et al. (1998)

_The Survey_

Views About Mathematics Survey
Form M 13

This survey is designed by the Modeling Instruction and ACEPT research teams at Arizona State University. It is intended to identify factors that affect people’s understanding of mathematics, and to assist in the design of instructional material.

Your participation is voluntary. The results will not affect your grade, even if you choose not to participate. All data are confidential. Your identity will not be disclosed to any party. Return of the survey materials will be considered your consent to participate.

Please:
- **Do not** write anything on this questionnaire.
- Mark your answers on the computer sheet.
- Use a **No. 2 pencil** only, and follow marking instructions on the computer sheet.
- Make **only one** mark per item.
- **Do not** skip any question.
- Avoid guessing. Your answers should reflect what you actually and honestly think.
- Plan to finish the survey in 30 minutes.
The example below illustrates the seven choices that you have for answering questions.

**Example**

Learning mathematics requires:

(a) a serious effort.

(b) a special talent.

What would each one of the eight choices mean?

1. Only (a), Never (b): Learning mathematics requires only a serious effort and no special talent at all.

2. Mostly (a), Rarely (b): Learning mathematics requires far more a serious effort than a special talent.

3. More (a) Than (b): Learning mathematics requires somewhat more a serious effort than a special talent.

4. Equally (a) & (b): Learning mathematics equally requires both a serious effort and a special talent.

5. More (b) Than (a): Learning mathematics requires somewhat more a special talent than a serious effort.

6. Mostly (b), Rarely (a): Learning mathematics requires far more a special talent than a serious effort.

7. Only (b), Never (a): Learning mathematics requires only a special talent and no serious effort at all.
1. Learning mathematics requires:
   (a) a serious effort.
   (b) a special talent.

2. If I had a choice:
   (a) I would never take another mathematics course.
   (b) I would still take mathematics for my own benefit.

3. **Reasoning skills** that are taught in mathematics courses can be helpful to me
   (a) in my everyday life.
   (b) if I were to major in mathematics or a related field.

4. My score on mathematics exams is a measure of how well:
   (a) I understand the covered material.
   (b) I can do things the way they are done by the teacher or in some course materials.

5. For me, doing well in mathematics courses depends on:
   (a) how much effort I put into studying.
   (b) how well the teacher explains things in class.

6. When I experience a difficulty while doing mathematics:
   (a) I quickly seek help, or give up trying.
   (b) I try hard to figure it out on my own.

7. When studying mathematics in a textbook or in course materials:
   (a) I memorize it the way it is presented.
   (b) I make sense of the material so that I can understand it.
8. Graphing calculators and computers:
   (a) bring new methods for solving mathematics problems.
   (b) speed up problem solving using established methods.

9. In mathematics, it is important for me to:
   (a) memorize technical terms and mathematical formulas.
   (b) understand the ideas and when and how to use them.

10. The purpose of mathematical formulas is to:
    (a) express meaningful relationships among variables.
    (b) provide ways to get numerical answers to problems.

11. After I go through mathematics text or course materials and feel that I understand them:
    (a) I can solve related problems on my own.
    (b) I have difficulty solving related problems.

12. The first thing I do when solving a word problem that involves mathematics is:
    (a) represent the situation with sketches and drawings.
    (b) search for formulas that relate givens to unknowns.

13. In order to solve a mathematics problem:
    (a) I need to have seen the solution to a similar problem before.
    (b) I apply general problem solving techniques.

14. Seeing alternate solutions to a mathematics problem is:
    (a) a waste of my time.
    (b) helpful for improving my problem solving abilities.
15. How well I do on mathematics exams depends on how well I can:
   (a) recall material in the way it was presented in class.
   (b) do tasks that are somewhat different from ones I have seen before.

16. Using graphing calculators or computers:
   (a) increases my interest in studying mathematics.
   (b) is a waste of my time.

17. Mathematical functions that represent relationships in the physical world are:
   (a) exact expressions of what is being represented.
   (b) approximate expressions of what is being represented.

18. The relationship among the sides of a right triangle expressed in the Pythagorean theorem is true because it has been:
   (a) proven by a logical argument.
   (b) verified by measurement.

19. Collecting and graphing real world data is useful for:
   (a) determining patterns and making general predictions.
   (b) obtaining numerical answers to specific problems.

20. For me, making unsuccessful attempts when solving a mathematics problem is:
   (a) a natural part of my pursuit of a solution to the problem.
   (b) an indication of my weaknesses in mathematics.

21. When solving a challenging mathematics problem, a mathematician:
   (a) makes many incorrect attempts.
   (b) moves directly to a correct solution.
22. The process of attempting to solve a problem that involves mathematical reasoning is:
   (a) a satisfying experience.
   (b) not a satisfying experience.

23. For me, the relationship of mathematics courses to everyday life is usually:
   (a) easy to recognize.
   (b) hard to recognize.

24. Proving a mathematical theorem requires:
   (a) evidence from the physical world.
   (b) a logically sound argument.

25. The role of a mathematics teacher is to:
   (a) show me how to work specific problems.
   (b) guide me in learning to solve problems.

26. The focus of student learning should be on:
   (a) memorizing specific content.
   (b) assimilating information from class and the text.

27. A student’s ability in mathematics is demonstrated by:
   (a) making logically sound arguments.
   (b) producing content knowledge.

28. A major goal of mathematics instruction is to:
   (a) impart information.
   (b) equip students to solve problems independently.

29. When completing a lesson in mathematics, I need to:
   (a) use only mathematical symbols.
   (b) write mathematics using words and mathematical symbols.
30. In solving mathematics problems, graphing calculators or computers help me:

(a) understand the underlying mathematical ideas.

(b) obtain numerical answers to problems.

Whole Class Results

The chart below (Figure 142) shows the number of students in each profile category before and after the instructional sequence.

Figure 142: Results of VAMS
Categories for each item

The ranges of responses constituting specific viewpoints are given in brackets after each item.

1. Learning mathematics requires:
   
   (a) a serious effort.

   (b) a special talent.

\{Expert View: options 1-3; Mixed View: options 4-5; Folk View: options 6-7\}

2. If I had a choice:
   
   (a) I would never take another mathematics course.

   (b) I would still take mathematics for my own benefit.

\{Expert View: options 6-7; Mixed View: options 4-5; Folk View: options 1-3\}

3. **Reasoning skills** that are taught in mathematics courses can be helpful to me
   
   (a) in my everyday life.

   (b) if I were to major in mathematics or a related field.

\{Item not assessed for Profile\}

4. My score on mathematics exams is a measure of how well:
   
   (a) I understand the covered material.

   (b) I can do things the way they are done by the teacher or in some course materials.

\{Expert View: options 1-4; Mixed View: option 5; Folk View: options 6-7\}

5. For me, doing well in mathematics courses depends on:
   
   (a) how much effort I put into studying.

   (b) how well the teacher explains things in class.

\{Expert View: options 1-3; Mixed View: option 4; Folk View: options 5-7\}
6. When I experience a difficulty while doing mathematics:
   
   (a) I quickly seek help, or give up trying.
   
   (b) I try hard to figure it out on my own.

   \{Expert View: options 6-7; Mixed View: option 5; Folk View: options 1-4\}

7. When studying mathematics in a textbook or in course materials:
   
   (a) I memorize it the way it is presented.
   
   (b) I make sense of the material so that I can understand it.

   \{Expert View: options 5-7; Mixed View: option 4; Folk View: options 1-3\}

8. Graphing calculators and computers:
   
   (a) bring new methods for solving mathematics problems.
   
   (b) speed up problem solving using established methods.

   \{Item not assessed for Profile\}

9. In mathematics, it is important for me to:
   
   (a) memorize technical terms and mathematical formulas.
   
   (b) understand the ideas and when and how to use them.

   \{Expert View: options 4-7; Mixed View: option 3; Folk View: options 1-2\}

10. The purpose of mathematical formulas is to:
    
   (a) express meaningful relationships among variables.
   
   (b) provide ways to get numerical answers to problems.

   \{Expert View: options 1-3; Mixed View: option 4; Folk View: options 5-7\}
11. After I go through mathematics text or course materials and feel that I understand them:
   
   (a) I can solve related problems on my own.
   
   (b) I have difficulty solving related problems.

   {Expert View: options 1-2; Mixed View: option 3; Folk View: options 4-7}

12. The first thing I do when solving a word problem that involves mathematics is:
   
   (a) represent the situation with sketches and drawings.
   
   (b) search for formulas that relate givens to unknowns.

   {Expert View: options 1-3; Mixed View: option 4; Folk View: options 5-7}

13. In order to solve a mathematics problem:
   
   (a) I need to have seen the solution to a similar problem before.
   
   (b) I apply general problem solving techniques.

   {Expert View: options 4-7; Mixed View: option 3; Folk View: options 1-2}

14. Seeing alternate solutions to a mathematics problem is:
   
   (a) a waste of my time.
   
   (b) helpful for improving my problem solving abilities.

   {Expert View: options 6-7; Mixed View: option 5; Folk View: options 1-4}

15. How well I do on mathematics exams depends on how well I can:
   
   (a) recall material in the way it was presented in class.
   
   (b) do tasks that are somewhat different from ones I have seen before.

   {Expert View: options 6-7; Mixed View: options 3-5; Folk View: options 1-2}
16. Using graphing calculators or computers:
   (a) increases my interest in studying mathematics.
   (b) is a waste of my time.

(Item not assessed for Profile)

17. Mathematical functions that represent relationships in the physical world are:
   (a) exact expressions of what is being represented.
   (b) approximate expressions of what is being represented.

(Expert View: options 6-7; Mixed View: option 5; Folk View: options 1-4)

18. The relationship among the sides of a right triangle expressed in the Pythagorean theorem is true because it has been:
   (a) proven by a logical argument.
   (b) verified by measurement.

(Expert View: options 1-2; Mixed View: options 3-4; Folk View: options 5-7)

19. Collecting and graphing real world data is useful for:
   (a) determining patterns and making general predictions.
   (b) obtaining numerical answers to specific problems.

(Expert View: options 1-3; Mixed View: options 4-5; Folk View: options 6-7)

20. For me, making unsuccessful attempts when solving a mathematics problem is:
   (a) a natural part of my pursuit of a solution to the problem.
   (b) an indication of my weaknesses in mathematics.

(Expert View: options 1-2; Mixed View: option 3; Folk View: options 4-7)
21. When solving a challenging mathematics problem, a mathematician:
   (a) makes many incorrect attempts.
   (b) moves directly to a correct solution.

   {Expert View: options 1-2; Mixed View: option 3; Folk View: options 4-7}

22. The process of attempting to solve a problem that involves mathematical reasoning is:
   (a) a satisfying experience.
   (b) not a satisfying experience.

   {Expert View: options 1-2; Mixed View: options 3-4; Folk View: options 5-7}

23. For me, the relationship of mathematics courses to everyday life is usually:
   (a) easy to recognize.
   (b) hard to recognize.

   {Expert View: options 1-4; Mixed View: option 5; Folk View: options 6-7}

24. Proving a mathematical theorem requires:
   (a) evidence from the physical world.
   (b) a logically sound argument.

   {Expert View: options 6-7; Mixed View: options 4-5; Folk View: options 1-3}

26. The role of a mathematics teacher is to:
   (a) show me how to work specific problems.
   (b) guide me in learning to solve problems.

   {Item not assessed for Profile}

30. The focus of student learning should be on:
   (a) memorizing specific content.
   (b) assimilating information from class and the text.

   {Item not assessed for Profile}
31. A student’s ability in mathematics is demonstrated by:
   (a) making logically sound arguments.
   (b) producing content knowledge.

\{Item not assessed for Profile\}

32. A major goal of mathematics instruction is to:
   (a) impart information.
   (b) equip students to solve problems independently.

\{Item not assessed for Profile\}

33. When completing a lesson in mathematics, I need to:
   (a) use only mathematical symbols.
   (b) write mathematics using words and mathematical symbols.

\{Item not assessed for Profile\}

30. In solving mathematics problems, graphing calculators or computers help me:
   (a) understand the underlying mathematical ideas.
   (b) obtain numerical answers to problems.

\{Item not assessed for Profile\}
Profiling Schema

The table below (Table 26) contains the schema used to categorize VAMS results, based on the response ranges for 22 of the items on the survey.

Table 26

Profiling Schema for VAMS

<table>
<thead>
<tr>
<th>PROFILE</th>
<th>Number of Expert Views</th>
<th>Number of Naïve Views</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expert</td>
<td>Greater than or equal to 12</td>
<td>Less than or equal to 8</td>
</tr>
<tr>
<td>Upper Transitional</td>
<td>Equal to 10</td>
<td>Less than or equal 10</td>
</tr>
<tr>
<td>Lower Transitional</td>
<td>Equal to 10</td>
<td>Equal to 9 or 10</td>
</tr>
<tr>
<td></td>
<td>Equal to 9</td>
<td>Less than or equal to 9</td>
</tr>
<tr>
<td></td>
<td>Equal to 8</td>
<td>Less than or equal to 8</td>
</tr>
<tr>
<td></td>
<td>Equal to 7</td>
<td>Less than or equal to 7</td>
</tr>
<tr>
<td>Naïve</td>
<td>Equal to 9</td>
<td>Greater than 9</td>
</tr>
<tr>
<td></td>
<td>Equal to 8</td>
<td>Greater than 8</td>
</tr>
<tr>
<td></td>
<td>Less than or equal to 7</td>
<td>Greater than 7</td>
</tr>
</tbody>
</table>
Up to (20 minus the number of expert views)
APPENDIX C

INSTRUCTIONAL MATERIALS
This appendix provides records of all the instructional materials developed and implemented during the instructional sequence. The first section provides a schedule of what materials were introduced for each class session. The second section provides all the problem sets and the last section provides the Extended Analysis Tasks (EATs).

**Schedule of Activity**

In this section I have provided a summary of the mathematical activity that took place for each of the 22 sessions of the instructional sequence.

**Class session #1**

- The Magic Square Problem

  In your group, create a 3x3 “Magic Square” that uses the numbers 1, 2, 3, 4, 5, 6, 7, 8, & 9.

- Pretest Administered

- VAMS administered

**Class session #2**

- Problem Set 1 Discussed

- The Dartboard Problem

  What is the highest score that you cannot get on the dartboard shown below if you may throw as many darts as you wish? Explain
• Fly-on-the-Wall Problem

Two walls and the ceiling of a room meet at right angles at the point P. A fly is in the air one foot from one wall, eight feet from the other wall, and nine feet from the point P. How many feet is the fly from the ceiling?

Class session #3

• Problem Set #2 discussed
• Began Catching up EAT

Person A sets out in a car going at 50 mph. Starting 3 hours later, person B tries to catch up. If person B goes at 75 mph, how long does it take to catch up?

Class session #4

• Problem Set #3 discussed
• Continued Catching up EAT

Class session #5

• Problem Set #4 discussed
• Inscribed Triangle Problem

You are given two intersecting straight lines and a point P marked on one of them as shown below. Show how to construct, using a straight edge and compass, a circle that is tangent through both lines and has the point P as one of the points of tangency to one of the lines.
Class session #6

- Problem Set #5 discussed
- The Triangle Problem

You are given a fixed triangle T with base B as below. Show that it is possible to construct, with a straightedge and compass, a straight line that is parallel to B and that divides T into two parts of equal area.

Class session #7

- Problem set #6 discussed
- Triangle Problem continued

Class session #8

- Problem Set 7 discussed
- Triangle Problem continued
Class session #9

- Problem Set 7 (continued)
- Triangle Problem Continued
- Inscribed Triangle Problem

Of all the triangles in a given circle with a fixed chord as base, which has the largest area?

Class session #10

- Problem Set 7 (continued)
- Evaporation Problem

A substance is 99% water. Some water evaporates, leaving a substance that is 98% water. How much of the water has evaporated?
Class session #11

- Problem Set 8
- The Rhombus Problem

In rhombus EFGH the coordinates of E and G are \((-6, -3)\) and \((2, 5)\) respectively. Find the area of the rhombus if the slope of segment EF is 2.

Class session #12

- Student presentations of the Evaporation Problem EAT

Class session #13

- Problem Set #9 discussed
- The Inequality Problems

Solve these both algebraically and graphically

\[
\frac{x + 6}{x + 1} < 2
\]

\[
\frac{3}{4} |x + 10| \geq 9
\]
Class session #14

- Problem Set 10 discussed
- The Recycling Problem

300 people were surveyed about their recycling habits.

35 people recycle paper and metal
40 people recycle metal and glass
60 people recycle paper and glass
90 people recycle paper
70 people recycle metal
105 people recycle glass
25 people recycle all three

Answer these questions.

1. How many people recycle paper only?
2. How many people recycle nothing?
3. How many recycle paper or glass?
4. How many recycle glass and metal, but not paper?

Class session #15

- Midterm Exam administered
Class session #16

- Student presentations of the Completing the Square EAT

Class session #17

- Problem Set #11 discussed
- The Triangle in Square Problem

Suppose ABCD is a square and CMN is an equilateral triangle. If the area of ABCD is 1 square unit, and if BM=ND, find the area of CMN
The Double Equilateral Triangle problem

In the figure, AD=DE=EF=EC=FD=FB. Prove that ΔABC is equilateral.

Class session #18

- Problem Set 12 discussed
- The Equidistant Points Problem

Find all the points on the line \(2x + 5y = 10\) which are equidistant from the coordinate axes

- The Box Problem

Given a sheet of 8.5 x 11 paper, your group wants to construct an open topped box by cutting squares out of the corners. You want to make a box with the greatest possible volume. What are the dimensions of such a box?
Class session #19

- Problem Set 13 discussed
- The System of Equations Problem

Solve the system of equations

\[
\begin{align*}
\frac{xy}{x + y} &= \frac{1}{2} \\
\frac{xz}{x + z} &= -\frac{1}{3} \\
\frac{yz}{y + z} &= \frac{1}{7}
\end{align*}
\]

Class session #20

- Problem Set 14 discussed
- The Round Trip Problem

Suppose that on a round trip, you travel at 30 mph on the way out and at 60 mph on the way back. What was your average speed?

Class session #21

- Problem Set 15 discussed
- First Box EAT discussed
- The Blackboard Problem

A set of consecutive integers beginning with 1 is written on a blackboard. One number is erased. The arithmetic mean of the remaining numbers is \(\frac{35}{17}\). What number was erased?
Class session #22

- Problem Set 16 discussed
- Second Box EAT discussed
- The Linear Factors Problems

1. For what integers \( n \) between 1 and 100 does \( x^2 + x - n \) factor into the product of two linear factors with integer coefficients?

2. Find an expression for the problem above that will find all integers \( n \) that will make \( x^2 + x - n \) factor into the product of two linear factors with integer coefficients.

Problem Sets

The following 16 problem sets were assigned as homework over the length of the instructional sequence.

Problem Sets

For the following problems, you need to:

- Solve the problem, show the work explain your reasoning
- Identify what mathematical concepts you used to solve the problem.
- Identify what problem solving strategies (if any) you used to solve the problem

1. How many different 9-person softball teams could be selected at random from a pool of 23 people?
2. An empty truck with driver weighs 4350 pounds. It is loaded with feed corn weighing 31 pounds per bushel. Between farm and market is a bridge with a 10,000-pound load limit. How many bushels can the truck legally carry?
3. A dog is tied outside to a ten-foot leash that is tied to the corner of a building in the shape of a regular pentagon. (Assume that the length of one side is greater than 10 feet.) How much area does the dog have in which to play?
4. An aerial photograph has a scale of 3 inches equals 20 miles. If the photograph is 12 inches by 18 inches, how many square miles are represented?
5. What is the largest sum of money – all in coins and no silver dollars – that you can have in your pocket without being able to make change for a dollar, a half-dollar, a quarter, a dime or a nickel?

6. For the following sequence made of toothpicks, let $S(n)$ represent the total number of toothpicks in the $n$th figure. Find the general term for the sequence.

![Diagram of toothpick figures]

**Problem Set 2**

For the following problems, you need to:

- Solve the problem, show the work and explain your reasoning
- Identify what mathematical concepts you used to solve the problem.
- Identify what problem solving strategies (if any) you used to solve the problem

1) Find the sum of the squares of the real roots of the equation $x^{256} - 256^{32} = 0$

2) A company estimates that in $t$ years the number of its employees will be $N(t)$, where $N(t) = 1000(0.8)^{t/2}$
   a) How many employees does the company expect to have in 4 years?
   b) At what rate is the number of employees expected to be changing at 4 years?

3) Lance has 2 different shaped unmarked buckets. One holds exactly 5 gallons, and one holds exactly 3 gallons. He needs to get exactly 4 gallons of water from a stream into one container. Explain how he can do this? Explain your reasoning

4) If $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$, then $a + b + c + d = ??$

5) Which leaves more open space – a round peg inside a square hole, or a square peg inside a round hole?

![Round and Square Pegs]

6) A box contains 2 pennies, 4 nickels and 6 dimes. Six coins are drawn without replacement, with each coin having an equal probability of being chosen. What is the probability that the value of the coins drawn is at least 50 cents?
Problem Set 3

For the following problems, you need to:

- Solve the problem, show the work and explain your reasoning
- Identify what mathematical concepts you used to solve the problem.
- Identify what problem solving strategies (if any) you used to solve the problem

1) Consider the function \( f(x) = \frac{cx}{2x + 3} \) where \( x \neq -\frac{3}{2} \). Find all values of \( c \) (if any) for which \( f(f(x)) = x \)

2) The tip of an arrow is to be in the shape of the figure bounded by a circle of radius \( \frac{1}{2} \) inch and two tangent lines to this circle drawn from a point 1 inch from the points of tangency. Find the area of arrow tip ABC. (The region shaded in light gray)

3) The perimeter of a rectangle is 200 feet.
   a) Let \( x \) express the width of the rectangle and write a function that expresses the area in terms of its width.
   b) Of all the possible rectangles with a perimeter of 200 ft., what are the measurements of the one that has the greatest area?

4) Hassan and Latifa earned the same amount of money, although one worked 6 days more than the other. If Hassan earned $36 a day and Latifa earned $60 a day, how many days did each work?

5) Two roads intersect at right angles. An eastbound car leaves the intersection traveling at a speed of 40 mph. Two hours later, a northbound car leaves the same intersection at a speed of 30 mph. How fast is the distance between the two cars changing 5 hours after the eastbound car leaves?

6) Write and solve a word problem that involves \( \sin x \).
Problem Set 4

For the following problems, you need to:

- Solve the problem, show the work and explain your reasoning
- Identify what mathematical concepts you used to solve the problem.
- Identify what problem solving strategies (if any) you used to solve the problem

1. Prove that \( \cot^2 x = \frac{1 - \sin^2 x}{\sin^2 x} \)

2. A certain farm allots $1000 for fencing a rectangular area that is to abut a highway. Because the fencing on the highway side must be attractive, it costs $4 per lineal foot. The other three sides of the area are fenced at $2 per foot. What are the dimensions of the rectangle that maximizes its area?

3. Studies show that for every 75¢ that women in the work force earn, men earn $1.00. Don claims that men make 25% more than women. Cindy claims men make 33% more than women. Who is correct and why?

4. The Bloom Company plans to market a new product – a pleasant smelling cat repellant that you can spray on your computer. Based on its market studies, Bloom estimates that it can sell 5500 units in 2001. The selling price is $2.00 per unit. Variable costs (cost of production) are estimated to be 40% of the selling price. Fixed costs are estimated to be $6000. How many units should she expect to sell before she can make a profit?

5. There are 20 people in a classroom. If every person in our class shakes hands with every other person in the class exactly once
   a. How many handshakes will have taken place?
   b. If there were 35 students in the class, how many handshakes would there be?
   c. What if there were \( n \) people?

6. The Great Wall of China is about 1500 miles long. The cross section is a trapezoid that is 25 feet tall, 25 feet wide at the bottom, and 15 feet wide at the top. How many cubic yards of material make up the Wall?

7. Write a word problem that will require the use of similar triangles to solve, and then solve it.
For the following problems, you need to:

- Solve the problem, show the work and explain your reasoning
- Identify what mathematical concepts you used to solve the problem.
- Identify what problem solving strategies (if any) you used to solve the problem

1. I have an elliptical shaped braided rug. The rug is 12 feet long and 6 feet wide. The rug is all brown except it has a red border that is 1 foot wide. What is the area of the red border? Can you find the solution without using the area formula for an ellipse?

2. The Mensa Store is having a special sale of used toys. Everybody is confused about the prices. At this sale, a train costs $22, a cart costs $17, a ball costs $17, but a truck is priced differently than the train. How much would a shovel cost at the Mensa Store?

3. Solve the following problems assuming you only have a balance scale at your disposal.
   a. Suppose you have 9 coins that are equal in appearance. Eight of these coins are identical in weight, and one is lighter. Explain how you can use the balance scale to identify the lighter coin by using the scale twice.
   b. Generalize this result. That is, if you have 3^n coins and all but one of them weigh the same; prove that you can find the odd coin by using the scale n times.

4. Compute $N$ so that $\frac{N}{x-5} + \frac{3}{x+4} = \frac{10x + 13}{x^2 - x - 20}$ for all $x \geq 2000$.

5. Points $A$, $B$, $C$, and $D$ lie on a line, in that order, with $AB = CD$ and $BC = 12, BE = CE = 10$. The perimeter of $\triangle AED$ is twice the perimeter of $\triangle BEC$. Find $AB$
Problem Set 6

For the following problems, you need to:

- Solve the problem, show the work and explain your reasoning
- Identify what mathematical concepts you used to solve the problem.
- Identify what problem solving strategies (if any) you used to solve the problem

1) Find the area of a rhombus that has one side of length 10, and diagonals that differ in length by 4.

2) Consider the family of quadratic functions given by $f(x) = (m^2 - m + 1)x^2 - (2m)x + 1$ where $m$ is any real number. For what values of $m$ will $f(x)$ always be positive?

3) Solve the following system of equations:
   \[\begin{align*}
   x^2 + y^2 &= 208 \\
   xy &= 96
   \end{align*}\]

4) Graph the following equation and compute the area it encloses.
   \[|2x - 10| + |5y - 10| = 20\]

5) On a separate sheet of paper, write a good word problem. Make enough copies for your classmates, and turn in 2 copies to me – one blank, and one with the problem solved.

Problem Set 7

1. Solve problems provided by your peers as assigned. You need only solve the problem – you do not need to list Resources and Heuristics for this assignment.
2. You will grade the problem you made up following the guidelines below.
   A. You will make up a rubric for grading your problem based on a 10-point scale.
   B. You will grade your problem using that rubric.
   C. You will write helpful comments on all papers.
   D. You will complete the grading before the next class session.
   E. You will turn the graded papers and the rubric into me.
   F. Your grade on this will be based on both the rubric and the grading.
Problem Set 8

(Note: you just need to solve the problems now – and of course, show your work and explain your reasoning)

1. How many different 5-card poker hands can be dealt from a deck of 52 cards?
2. What is the probability of being dealt a royal flush? (i.e., ten, jack, queen, king and ace of the same suit.)
3. For $x > 0$, what is the smallest value of $x + \frac{5}{x}$. Solve this problem two different ways.
4. When an integer is divided by 15, the remainder is 7. Find the sum of the remainders when the same integer is divided by 3 and by 5.
5. Apple weighed her 4 dogs on pairs. Together, Hilbert and Euclid weighed 110 lbs, Euclid and Gauss weighed 103 lbs, and Gauss and LaGrange weighed 108 lbs. How many pounds would Hilbert and LaGrange weigh together?
6. Answer the following:
   a. Prove the following: If two numbers are divisible by 4, then their average is divisible by 2.
   b. Write a generalized version of the statement in part (a).
   c. Prove the statement you wrote in part (b).

Problem Set 9

1. The rate of decay of a radioactive substance is proportional to the amount of substance present at any time $t$. In 1840 there were 50 grams of the substance and in 1910 there were 35 grams. To the nearest gram, how many grams of the substance remain in 1990?
2. How many connecting cables are needed in order that any two of nine offices in a building can communicate directly.
3. Generalize #2. That is find how many cables are needed if there are \( n \) offices.

4. A small corporation borrowed $500,000 to expand its product line. Some of the money was borrowed at 9\%, some at 10\% and some at 12\%. How much was borrowed at each rate if the annual interest was $52,000 and the amount borrowed at 10\% was two and a half times the amount borrowed at 9\%?

5. A builder wishes to construct a ramp that is 24 feet long, and which rises to a height of 5 feet above the level ground. Approximate the angle that the ramp should make with the level ground.

6. In the adjoining figure, \( AB \) and \( BC \) are adjacent sides of square \( ABCD \); \( M \) is the midpoint of \( AB \); \( N \) is the midpoint of \( BC \); \( AN \) and \( CM \) intersect at \( O \). Find the ratio of the area of \( AOCD \) to the area of \( ABCD \)

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**Problem Set 10**

1. Consider the pattern below.
   a. How many congruent triangles will be in the 7\(^{th} \) figure?
   b. Find an expression for the \( n \)th figure.

2. The roots of \( x^2 - ax + b = 0 \) are the squares of the roots of \( x^2 - cx + d = 0 \). Express \( a \) in terms of \( c \) and \( d \).

3. The gauge of an oil tank indicated that it was \( \frac{1}{7} \) full. After 240 gallons were added, the gauge indicated that the tank was now \( \frac{4}{7} \) full. How many gallons does the tank hold? (Assuming that the gauge is accurate)
4. A certain water lily grew extremely quickly and it doubled its surface area at the end of each day. At the end of the 30th day, it had entirely covered the pond in which it lived. If a second lily, identical to the first, has been in the pond, how long would the two lilies have taken to cover the entire pond?

5. For what values of k are the roots of \( 2x^2 - 3x + k = 0 \) equal?

6. The measure of the largest angle of a triangle is 10° more than 4 times the smallest angle. The sum of the smallest and the largest angles is three times the measure of the other angle. Find the measure of each angle in the triangle.

**Problem Set 11**

1. After the game, an excited coach randomly distributes the jackets to the six girls who own them. What is the probability that all the girls receive the correct warm up jacket?

2. The sum of the first 5 terms of an arithmetic sequence is 40, and the sum of the first 10 terms of the sequence is 155. Find the formula for the \( n \)th term of the sequence.

3. Solve \( \ln x + \ln(x - 2) = 1 \)

4. Find 2 numbers whose sum is 10, such that the sum of the square of one number plus 4 times the other number is a minimum

5. In the isosceles trapezoid \( ABCD \), with \( AB \parallel CD \), and diagonals \( AC \perp BD \). If \( AB = 4 \) and \( AD = 7 \), compute \( CD \).

\[
\begin{array}{c}
A \quad 4 \quad B \\
7 \quad D \quad C
\end{array}
\]

6. WITHOUT A CALCULATOR rank the following from greatest to least: \( e^e, \pi^\pi, e^\pi, \pi^e \) and explain your reasoning.
Problem Set 12

1. Consider the letters that make up APPLE.
   a. How many distinct ways are there to arrange these letters?
   b. If I out all these arrangements of these letters in a hat, what is the probability that I will chose the one with APPLE on it?
2. Solve $\left| \frac{x - 2}{x + 3} \right| < 4$
3. Solve the following system:
   
   \[
   \begin{align*}
   4x + 4y &= 10 \\
   2x + 2y &= 5xy
   \end{align*}
   \]

4. Sparky owns a used car lot. Yesterday he sold 2 cars for $7,500 a piece. He made 25% profit on the first car, but the second sale resulted in a 25% loss. Did Sparky make money, break even or lose money?
5. What is the value of the following:
   \[2 - 4 + 6 - 8 + 10 - 12 + 14 - \cdots - 100\]

6. Solve $e^{2x} - 3e^x = -2$
7. In Trapezoid $ABCD$ with bases $AB$ and $CD$, we have $AB = 52$, $BC = 12$, $CD = 39$ and $DA = 5$. Find the area of $ABCD$.

Problem Set 13

1. The product of the ages of a group of teenagers is 10,584,000. Find the number of teenagers in the group and the sum of their ages.
2. Solve for $x$: $(\log x)^2 = \log(x^2)$. 
3. A drawer contains two red and 3 blue socks. Socks are taken out of the drawer successively without replacement. What is the probability that the fourth sock drawn is blue?

4. Select any odd number. Square it and subtract 1. Prove that the result will always be divisible by 8.

5. The altitudes of a parallelogram have lengths 8 and 10 and intersect at an angle whose sine is $\frac{1}{4}$. Compute the area of the parallelogram.

6. For all real $x$ where $f(2x) = x^2 - x + 3$, express $f(x)$ in terms of $x$

Problem Set 14

1. Let $f$ be the function defined by $f(n) = 2f(n - 1) + 3f(n - 2)$ where $f(1) = 1$ and $f(2) = 2$. Compute $f(5)$.

2. Three fair dice are tossed (all faces have the same probability of coming up). What is the probability that the three numbers facing up can be arranged to form an arithmetic progression with a common difference of 1?

3. Show that the product of three consecutive integers, the smallest which is even, is divisible by 24.

4. Find the result when the sum of the first eighty positive odd integers is subtracted from the first eighty even integers.

5. The square below is divided into 5 congruent rectangles. If the perimeter of one of the rectangles is 30 units, how many units is the perimeter of the square?

\[
\begin{array}{c}
\hspace{1cm}
\end{array}
\]

Problem Set 15

1. A surveyor wishes to measure the height of a tall tree and he does so by measuring a distance of 100 feet from the base of the tree (the ground is level). Then he observes that the angle of inclination from that spot to the top of the tree is $38^\circ$. How high is the tree? How far is the surveyor from the top of the tree?

2. Let $f$ be a differentiable function defined for all real numbers $x$, with the following properties:
   
i. $f'(x) = ax^2 + bx$
ii. \( f'(t) = 6 \) and \( f''(t) = 18 \)
iii. \( \int_1^2 f(x) \, dx = 18 \)

Find \( f(x) \).

3. The management of a factory has found that the maximum number of units a worker can produce in a day is 50. The learning curve for the number of units \( N \) produced per day after a new employee has worked \( t \) days is given by \( N = 50(1 - e^{-kt}) \). After 20 days on the job, Steve produced 31 units in 1 day. How many days should pass before Steve is producing 45 units a day?

4. Diane is preparing her college schedule for next semester. She needs to choose three courses. She may select one of 5 math courses, one of 3 science courses, one of 8 courses from the social sciences and humanities. In how many ways can she select her courses?

5. If \( \theta \) is an acute angle and \( \sin 2\theta = a \), find \( \sin \theta + \cos \theta \).

**Problem Set 16**

1. A freight train one kilometer long goes through a tunnel that is one kilometer long. If the train is traveling at a speed of 15 kilometers per hour, how long does it take the train to pass through the tunnel?

2. Mendel found that snapdragons have no color dominance: a snapdragon with one red gene and one white gene will have pink flowers. If a pure red snapdragon is crossed with a pure white snapdragon, what is the probability of red flowers on the offspring?

3. The diameter of the earth is approximately 7,920 miles. The diameter of Jupiter is approximately 88,640 miles. How many earths could fit inside one Jupiter?

4. The Arizona Fish and Game Commission releases 100 deer into a wilderness area. The commission believes that the carrying capacity of the area is 500 deer, and the growth of the herd can be modeled by the logistics curve
   \[ P = \frac{500}{1 + 4e^{-kt}}, \quad 0 \leq t \text{ where } P \text{ is the herd size and } t \text{ is the time measured in years.} \]
   Find \( k \) if the herd size is 170 after 2 years.

5. Given: All Zips are Zowies, but only some Zowies are Zoids. Which of the following statements are true?
   a. No Zips can be Zoids.
   b. If something is not a Zowie, then it can’t be a Zip.
c. If something is a Ziod, then it can’t be a Zip.

6. A piece of wire is 52 feet long. It is cut into 2 pieces, each of which is then bent into a square. The total area of the 2 squares is 97 square feet. What are the lengths of the sides of the two squares?

**Problem Set 17**

1. Suppose you have the letters $a, b, c, d,$ and $e$.
   a. In how many arrangements can you make of these letters?
   b. How many of them will have $a$ followed immediately by $b$? Explain.
2. If water is being drained from a swimming pool and $V$ represents the volume of the water in the pool $t$ minutes after the draining starts, and
   \[ V = 250(1600 - 80t + t^2) \]
   find out how fast the water is draining at $t = 5$ minutes.
3. Amanda noticed that if she took her apartment number and increased it by 6, the result was a perfect square. She also noticed that if she decreased her apartment number by 6, the result was the square root of that perfect square. What is Amanda’s apartment number?
4. Find all real $x$ such that $\sqrt{1 - \sqrt{1 - x}} = x$
5. Quadrilateral MATH contains right angles at vertices A and H. If $\angle AMH$ measures $120^\circ$, $MA=10$ and $MH=40$, find the length of $TH$.

6. A $3\times3$ magic square consists of 9 consecutive positive integers. Create such a square so that the common row, column and diagonal sum is equal to 30.
**Extended Analysis Tasks**

This section provides the student versions of all five Extended Analysis Tasks (EAT’s) that were included in the instructional sequence.

*EAT #1: The Catching up Problem*

An example illustrating Extended Analysis

Many problems in high school mathematics are "numbers in - numbers out" problems. Although the techniques used to solve the problems may involve sophisticated ideas from algebra, analysis, trigonometry, etc., the net result is simply a numerical answer based on numbers given in the problem statement.

The purpose of this exercise is to demonstrate how problems of this sort can be generalized in a systematic way: if the numbers given in the problem statement are replaced by general parameters, the result of the analysis gives the answer as a function of these parameters.

Here is a simple high school level problem that we will use to illustrate this idea:

Person A sets out in a car going at 50 mph. Starting 3 hours later, person B tries to catch up. If person B goes at 75 mph, how long does it take to catch up?

Solve the problem. Show all your work and be prepared to share your reasoning and discuss what problems high school students have with this problem.

This type of problem can be useful in giving students practice in setting up and solving equations. Still, much richer mathematics is involved in taking the problem one step further to give a general answer to a general problem.

As a start on the process of generalizing, suppose we replace person B’s speed of 75 mph with a parameter: the speed \( w \).

1. Before reading on, solve the problem for \( t \) in terms of the speed \( w \) of person B.
2. Sketch a graph of this function. What does it tell you? How is it interesting?
This is not the only way to generalize the problem. To get a fuller picture, we make a table of the relevant parameters in the problem:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Original value</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed of person A</td>
<td>$v$</td>
<td>50 mph</td>
</tr>
<tr>
<td>speed of person B</td>
<td>$w$</td>
<td>75 mph</td>
</tr>
<tr>
<td>speed increase (how much faster B is than A)</td>
<td>$\Delta v$</td>
<td>25 mph</td>
</tr>
<tr>
<td>head start time (head start A has over B)</td>
<td>$T$</td>
<td>3 hours</td>
</tr>
<tr>
<td>catch-up time (time for B to catch up with A)</td>
<td>$t$</td>
<td>6 hours</td>
</tr>
</tbody>
</table>

The function $f$ we have derived above gives $t$ as a function of $w$, keeping the other parameters with their original numerical values. As another way to generalize, we can leave the catch-up speed at 75 mph, and ask for the catch-up time $t$ as a function of the "head start" time $T$.

1. Do you think $t$ will be a linear function of $T$ or not? Explain

2. Now find $t$ explicitly as a function of $T$.

3. What is the catch-up time $t$ as a function of the difference $\Delta v$ of speeds?

4. Write a statement or theorem that summarizes what you discovered in this generalization exercise.

5. Write an isomorphic problem and show that it has the same mathematical structure as the original problem.

_EAT #2: The Evaporation Problem_

1. Solve the Evaporation Problem. Show work and explain your reasoning.

A substance is 99% water. Some water evaporates, leaving a substance that is 98% water. How much of the water has evaporated?

2. Totally parametrize the problem
3. Extend the problem in some way that is interesting and reveals something about the mathematics that you didn't know. This might include the following, or anything else you think is illuminating.

- Looking at extreme cases.
- Modeling the function in some way.
- Changing some aspect of the problem and discerning the resulting effect.
- Graphing the function in at least one way.
- Contrasting functions with respect to different variables.
- Exploring dimensional analysis.
- Looking for isomorphic problem situations.

4. Create an isomorphic problem situation and show that it has the same mathematical structure as the original problem

\[ f(x) = 6x^2 + 5x - 6 \]

\[ EAT \#3: \text{The Completing the Square Problem} \]

For the function \( f(x) = 6x^2 + 5x - 6 \), complete the square to name the vertex of the function as well as the x intercepts.

1. Parameterize this problem and solve it in a generalized form. (i.e. replace the numbers with letters and solve the problem again)
2. Extend the problem in some way that is interesting and reveals something about the mathematics that you didn’t know. This might include the following, or anything else you think is illuminating.

- Looking at extreme cases.
- Modeling the function in some way.
- Changing some aspect of the problem and discerning the resulting effect.
- Graphing the function in at least one way.
- Contrasting functions with respect to different variables.
- Exploring dimensional analysis.

3. Make sure you come to some conclusion as a result of your exploration

\[ EAT \#4: \text{The First Box Extension} \]

Recall the Box Problem solved in class:
Given a sheet of 8.5 x 11 paper, your group wants to construct an open topped box by cutting squares out of the corners. You want to make a box with the greatest possible volume. What are the dimensions of such a box?

1. Suppose the rectangular paper has length $L$ and width $I$ with square cut out with side $x$. Find $V(x, L)$ and then find the $x(L)$ (that is, $x$ in terms of $L$) that produces the maximum volume.

2. Graph $x(L)$ and interpret your results in the context of the problem.

3. Locate the points on the graph that represent the original problem (above) and the square (as we did in class).

**EAT #5: The Second Box Extension**

**More Fun with the Box Problem:**

1. If the critical points for the function $V = x(1 - 2x)(L - 2x)$ are $x(L) = \frac{L + 1 \pm \sqrt{L^2 - L + 1}}{6}$, determine which is the maximum and which is the minimum. (Recall that $V$ is a cubic function) Explain your reasoning.

2. What does the graph of $x(L)_{max}$ tell us about the problem?

3. Find $\lim_{L \to \infty} x(L)_{max}$ and interpret the meaning of that limit.

4. Suppose I have a fixed length of fence that I want to use to create a rectangular shaped pen along the side of a barn. If I call the fixed length “1”, what should the dimensions of the pen be?

5. How does the problem in #4 relate to the box problem? (hint – think crosssection)

6. Consider the problems we have already solved. (The $8\frac{1}{2} \times 11$ sheet and the $11 \times 11$) and compare the area of the folded up region with the base of the box. Can we generalize the finding to the $1 \times L$ sheet? Convince me one way or the other.

**Final Project EAT**

Final Exam: Take-Home Portion

You will select a math problem from a high school math book. You will solve the problem, and then you will extend the problem in some way that is interesting and reveals
something about the mathematics that you didn’t already know. This might include several of the following, or anything else you think is illuminating.

- Looking at extreme cases.
- Modeling the function in some way.
- Changing some aspect of the problem and discerning the resulting effect.
- Graphing the function in at least one way.
- Contrasting functions with respect to different variables.
- Exploring dimensional analysis.

- Then you will draw a conclusion from your exploration. This will be worth 40% of your final exam grade.