Review Test 1

1. First-order HODEs/IVPs. Use separation of variables.
   (a) \( y' + y = 0 \) (Ans: \( y(t) = Ce^{-t} \))
   (b) \( y' + 2y = 0 \) with \( y(0) = 3 \) (Ans: \( y(t) = 3e^{-2t} \))

2. First-order ODEs/IVPs. Find general solution as sum of general solution of HODE (as in 1) and a particular solution of ODE (by inspection), or using variation of the constant appearing in general solution of HODE. Impose IC last.
   (a) \( y' + 2y = 4 \) with \( y(0) = 3 \) (Ans: \( y(t) = 2 + e^{-2t} \))
   (b) \( y' + 2y = 2t \) with \( y(0) = 3 \) (Ans: \( y(t) = t - \frac{1}{2} + \frac{7}{2} e^{-2t} \))
   (c) \( y' + 2y = 2e^{-4t} \) with \( y(0) = 3 \) (Ans: \( y(t) = -e^{-4t} + 4e^{-2t} \))
   (d) \( y' + 2y + 6e^{-4t} = 0 \) with \( y(0) = 3 \) (Ans: \( y(t) = 3e^{-4t} \))
   (e) \( y' + 2y = e^{-2t} \) with \( y(0) = 3 \) (Ans: \( y(t) = (t + 3)e^{-2t} \))
   (f) \( y' + 2y = \sin t \) with \( y(0) = 3 \) (Ans: \( y(t) = -\frac{1}{5} \cos t + \frac{2}{5} \sin t + \frac{16}{5} e^{-2t} \))
   (g) \( y' + iy = \sin t \) with \( y(0) = -\frac{1}{2} \) (Ans: \( y(t) = -\frac{1}{2} \cos t + \frac{1}{2} te^{-it} \))

3. Second-order HODEs/IVPs with constant coefficients. Determine characteristic equation and type of solution according to roots.
   (a) \( y'' + y = 0 \) with \( y(0) = 3 \) and \( y'(0) = 0 \) (Ans: \( y(t) = 3 \cos t \))
   (b) \( y'' + y = 0 \) with \( y(0) = 3 \) and \( y'(0) = -3 \). Write the solution as \( y(t) = A \sin(\omega t + \varphi) \) with \( A > 0 \). What are \( A \), \( \omega \) and \( \varphi \)? (Ans: \( y(t) = 3 \cos t - 3 \sin t = 3 \sqrt{2} \sin(t + \frac{3\pi}{4}) \))
   (c) \( y'' + 2y' + y = 0 \) with \( y(0) = 3 \) and \( y'(0) = 0 \) (Ans: \( y(t) = 3e^{-t} \))
   (d) \( y'' + 3y' + 2y = 0 \) with \( y(0) = 3 \) and \( y'(0) = 0 \) (Ans: \( y(t) = 6e^{-t} - 3e^{-2t} \))
   (e) \( y'' + y' - 2y = 0 \) with \( y(0) = 3 \) and \( y'(0) = a \). For what value(s) of \( a \) does the solution remain bounded as \( t \to \infty \)? (Ans: \( a = -6 \))
   (f) \( 5y'' + \gamma y' + 3y = 0 \). For what minimal value of \( \gamma \) does the solution tend to 0 as \( t \to \infty \) without oscillating? (Ans: \( \gamma = 2\sqrt{15} \))
   (g) \( y'' + 2y' + 2y = 0 \). Determine both complex and real forms of the general solution (Ans: \( y(t) = Ce^{(-1+i)t} + C_2e^{(-1-i)t} = e^{-t}(C_3 \cos t + C_4 \sin t) \))

4. Reduction of second-order ODE/IVP to system of first-order ODEs/IVPs. Use \( z = y' - ry \).
   (a) \( y'' + 2y' + y = 0 \) with \( r = 0 \), \( r = -1 \)
   (b) \( y'' + 2y' - 3y = 0 \), \( y(0) = 3 \), \( y'(0) = -1 \), with \( r = 0 \), \( r = 1 \), \( r = -3 \)

5. Second-order ODEs/IVPs via sequential first-order ODEs/IVPs. Use \( z = y' - ry \) where \( r \) is a root of the characteristic equation.
   (a) \( y'' + 2y' + y = t \), \( y(0) = 3 \), \( y'(0) = 0 \), with \( r = -1 \) (Ans: \( y(t) = t - 2 + (4t + 5)e^{-t} \))
   (b) \( y'' + 2y' - 3y = 8e^t \), \( y(0) = 3 \), \( y'(0) = 1 \), with \( r = 1 \), \( r = -3 \) (Ans: \( y(t) = e^{-3t} + 2(t + 1)e^t \))
   (c) Reduce \( y'' - 2y' + y = g(t) \) to a system of two first-order ODEs, one of which is \( y' = y + z \) (Ans: the other ODE is \( z' = z + g(t) \))
6. Undetermined coefficients

(a) Determine a constant $C$ such that $y(t) = Cte^t$ satisfies the ODE $y'' + 2y' - 3y = 4e^t$.  
    (Ans: $C = 1$)

(b) Determine constants $a$ and $b$ such that $y(t) = a\cos 3t + b\sin 3t$ satisfies the ODE $\quad y'' + 2y' - 3y = 6\cos 3t$ (Ans: $a = \frac{1}{5}, b = -\frac{2}{3}$)

7. MATLAB programming. Function, arrays, ode23 calling sequence, coefficientwise operations

(a) Write a two line MATLAB function file implementing the function $g(t) = te^{-t}$. Write a statement to obtain $g(2)$ in the Command Window. (Ans: function $\ y = \text{ode23}([0 \ 20], 2, [], a)$

(b) A MATLAB file includes the following statements
   
   $[t, y] = \text{ode23}([0 \ 20], 2, [], a)$;
   
   function dy/dt = f(t, y, a)
   
   dy/dt = y + a*t;
   
   What is the IVP solved? (Ans: $y = y + at$, $y(0) = 2$)

(c) What is the dimension of the array $y$ resulting from the MATLAB statement

   $[t, y] = \text{ode23}([0\ 20], 0.1:5, [2; -1], [], a)$;

   (Ans: $51 \times 2$)

(d) What is the dimension of the array $y$ resulting from the MATLAB statement

   $[t, y] = \text{ode23}([0\ 20], 0.5, [2; -1], [], a)$;

   (Ans: $n \times 2$, $n$ determined by ode23)

(e) A MATLAB array $y$ has dimension $101 \times 3$. What is a command for plotting the third column of $y$ versus the first one? (Ans: $\text{plot}(y(:, 1), y(:, 3))$)

(f) A MATLAB program includes the statements $X = [2, 3; 4, 5; 6, 7]; \ Y = [4; 6; -3];$

   What is the result of the commands $X(:, 1) * Y$? $X(:, 1) * Y$? (Ans: $\begin{bmatrix} 8 \\ -18 \end{bmatrix}$, the second command produces an error message)

8. Complex numbers, Euler's formula

(a) Transform the first-order complex IVP $y' + iy = e^t$, $y(0) = 2 + i$, into a first-order real IVP involving a system of two coupled ODEs (hint: assume $y = y_1 + iy_2$ with $y_1$ and $y_2$ real and identify real and imaginary parts). (Ans: $\begin{cases} \frac{d}{dt} y_1 = y_2 + \cos t \\ \frac{d}{dt} y_2 = -y_1 + \sin t \end{cases}$ $y_1(0) = 2$ $y_2(0) = 1$)

(b) Reduce the expression $y(t) = \frac{1}{1+i} e^{(1+i)t} + \frac{1}{1-i} e^{(1-i)t}$ to a real function. (Ans: $y(t) = e^t(\cos t + \sin t)$)

(c) Use Euler's formula to determine an expression of $\sin 3t$ in terms of $\cos t$ and $\sin t$ (Ans: $\sin 3t = 3\sin t \cos^2 t - \sin^3 t$).

9. Theory

(a) Let $y_1$ and $y_2$ be two solutions of $y'' + a(t)y' + b(t)y = 0$. Show that $y = C_1 y_1 + C_2 y_2$ is also a solution of $y'' + a(t)y' + b(t)y = 0$ for any choice of constants $C_1$ and $C_2$.

(b) Let $y_{\text{HODE}}$ be a solution of $y'' + a(t)y' + b(t)y = 0$ and $y_{\text{part}}$ be a solution of $y'' + a(t)y' + b(t)y = g(t)$. Show that $y = y_{\text{HODE}} + y_{\text{part}}$ is also a solution of $y'' + a(t)y' + b(t)y = g(t)$.

(c) Let $y_1$ be a solution of $y'' + a(t)y' + b(t)y = 0$, $y(0) = a$, $y'(0) = b$, and $y_2$ be a solution of $y'' + a(t)y' + b(t)y = g(t)$, $y(0) = y'(0) = 0$. Show that $y = y_1 + y_2$ is a solution of $y'' + a(t)y' + b(t)y = g(t)$, $y(0) = a$, $y'(0) = b$. 